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A VARIATIONAL PRINCIPLE FOR PLASTIC MATERIALS WITH STRAIN-HARDENING[1]

By P. Hodge and W. Prager

1. Introduction. This paper is concerned with a minimum principle for the stresses in a plastic material with strain-hardening. Only such states of stress (or strain) are considered which can be reached from the natural state by a single loading followed by at most one (complete or partial) unloading.

Handelman, Lin, and Prager[2] discussed both finite and differential stress-strain relations which might be used under these conditions. These relations assume different forms depending on whether "loading" or "unloading" takes place. In the case of simple tension or pure shear, the distinction between loading and unloading is obvious. However, in more general cases where some stress components may be decreasing while others are increasing, it becomes necessary to establish a formal mathematical criterion for loading and unloading. To this end, a convenient function f of the stress components may be used, and an infinitesimal change of stress may be said to constitute loading, unloading, or a neutral change according to whether the corresponding differential df is positive, negative, or zero.

It would seem reasonable to expect the stress-strain relation for neutral change of stress to assume the same form, whether this change of stress is considered as a limiting case of loading or unloading. Handelman, Lin, and Prager proved, however, that no finite stress-strain relation will fulfil this expectation. In view of this fact the general validity of finite stress-strain relations appears doubtful.

It is also worth noting that any finite stress-strain relation may be differentiated and thus put in the form of a differential relation. However, the converse is not true since, in general, a differential relation may not be integrated to yield a finite relation.

In view of these facts the following discussion will be restricted to stress-strain relations of the differential form. In the customary tensor notation, we denote the strain by ϵ_{ij}, the reduced stress (i.e. the stress divided by Young's modulus) by σ_{ij}, and Poisson's ratio by ν. Using the summation convention regarding repeated subscripts, we define the *reduced mean normal stress* as

$$s = \tfrac{1}{3}\sigma_{ii} \tag{1}$$

and the *reduced*[3] *stress deviation* as

$$s_{ij} = \sigma_{ij} - s\delta_{ij}, \tag{2}$$

[1] The results presented in this paper were obtained in the course of research conducted under Contract N7onr-358 sponsored jointly by the Office of Naval Research and the Bureau of Ships.

[2] G. H. Handelman, C. C. Lin, and W. Prager, Q. Appl. Math. 4, 397–407 (1947).

[3] For the sake of brevity, the terms "stress" or "stress deviation" will be used in the following to denote "reduced stress" or "reduced stress deviation".

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where δ_{ij} is the Kronecker delta. We further use the notations

$$J_2 = \tfrac{1}{2}s_{ij}s_{ij}, \qquad J_3 = \tfrac{1}{3}s_{ij}s_{jk}s_{ki}, \tag{3}$$

and

$$t_{ij} = s_{ik}s_{kj} - \tfrac{2}{3}J_2\delta_{ij}. \tag{4}$$

Handelman, Lin, and Prager used the differential dJ_2 as the criterion for loading. However, considerably greater generality may be obtained by using the differential of a function $f(J_2, J_3^2)$ which is characteristic for the material under consideration. With this minor change, the differential stress-strain relations of Handelman, Lin, and Prager become

$$\left.\begin{aligned} de_{ij} &= (1+\nu)ds_{ij} + (1-2\nu)ds\delta_{ij} \\ &\quad + [p(J_2, J_3^2)s_{ij} + q(J_2, J_3^2)J_3 t_{ij}]df \quad \text{when } df \geq 0, \\ de_{ij} &= (1+\nu)ds_{ij} + (1-2\nu)ds\delta_{ij} \quad \text{when } df \leq 0. \end{aligned}\right\} \tag{5}$$

These two equations may be combined and written as follows:

$$de_{ij} = (1+\nu)ds_{ij} + (1-2\nu)ds\delta_{ij} + \tfrac{1}{2}[ps_{ij} + qJ_3 t_{ij}][df + |\,df\,|]. \tag{6}$$

Here the first two terms on the right-hand side represent the elastic change of strain associated with the change of stress under consideration, while the last term represents the change in permanent strain.

The functions p, q, and f, however, are not entirely arbitrary. Let us consider the surfaces $f = \text{const.}$ in a five-dimensional space with the five independent stress deviation components as coordinates (there are only five such components, since $s_{ii} = 0$). Unless these surfaces are convex with respect to the origin, a ray from the origin may intersect a surface $f = \text{const.}$ more than once. This would imply that a process during which the stresses are continually increased in such a manner that all ratios between the components of the stress deviation remain constant, would be classified first as loading, then as unloading. This is obviously absurd. A sufficient condition for the surfaces $f = \text{const.}$ to be convex is that f be a positive definite form of the components of the stress deviation.

Next, we take account of the irreversible character of plastic deformation. Work done in producing plastic deformations cannot be reclaimed. Defining the *strain deviation* as

$$e_{ij} = \epsilon_{ij} - \tfrac{1}{3}\epsilon_{kk}\delta_{ij}, \tag{7}$$

we write the condition of irreversibility in the form

$$s_{ij}de_{ij}^{(1)} > 0, \tag{8}$$

where $de_{ij}^{(1)}$ is the change in permanent strain represented by the third term on the right hand side of (6). Thus, (8) is equivalent to

$$2pJ_2 + 3qJ_3^2 > 0. \tag{9}$$

Finally, we demand that when the state of stress σ_{ij} throughout the body and the infinitesimal increments $dT_j = d\sigma_{ij}n_i$ of the surface stresses are given, we

shall have uniquely determined increments of stress throughout the body. Melan[4] has shown that as a consequence of this requirement, p and q are determined to within an arbitrary positive definite invariant factor if f is given. Experimentally, however, it is much easier to determine p and q than f,[5] We shall therefore, prove the converse of Melan's theorem, in Sec. 2, establishing the fact that f is determined by p and q. Results analoguous to the reciprocity theorem of Maxwell and Betti and to the principle of virtual displacements will be established in Sec. 3. In Sec. 4 it will be proved that the stress-strain relation obtained in Sec. 2 satisfies all conditions formulated above. Section 5 is concerned with a minimum principle for the case where the given rates of stress at the surface produce loading throughout the body and Sec. 6 with the extension of this principle to other cases. Finally in Sec. 7 we shall discuss the limitations of the theory.

2. Determination of $d\epsilon_{ij}$ in terms of f. If we are given any two well-behaved functions $p(J_2, J_3^2)$ and $q(J_2, J_3^2)$, there exists an integrating factor $F(J_2, J_3^2)$, unique to within a multiplicative constant, such that

$$[2pdJ_2 + qd(J_3^2)]/2F \tag{10}$$

is a perfect differential. We shall demand that p and q be such that F be positive definite, and designate the resulting perfect differential by df.[6] Furthermore, we shall demand that the resulting function f be a positive definite form in the components of the stress deviation. Then

$$p = F\partial f/\partial J_2, \qquad q = 2F\partial f/\partial J_3^2. \tag{11}$$

From the definitions (1) to (4) it follows that

$$\partial J_2/\partial\sigma_{ij} = s_{ij}, \qquad \partial J_3/\partial\sigma_{ij} = t_{ij}. \tag{12}$$

In view of (11) and (12), Eq. (6) now becomes

$$d\epsilon_{ij} = (1+\nu)ds_{ij} + (1-2\nu)ds\delta_{ij} + \tfrac{1}{2}F(\partial f/\partial\sigma_{ij})(df + |df|). \tag{13}$$

If the elastic part of the strain is denoted by

$$\epsilon_{ij}^{(0)} = (1+\nu)s_{ij} + (1-2\nu)s\delta_{ij}, \tag{14}$$

the stress-strain relation can finally be written as

$$\dot\epsilon_{ij} = \dot\epsilon_{ij}^{(0)} + \tfrac{1}{2}F(\partial f/\partial\sigma_{ij})(\dot f + |\dot f|), \tag{15}$$

where, for convenience, time derivatives have been used instead of differentials.

3. Some inequalities, reciprocity theorems, and an analogy to the principle of virtual displacements. Let us consider a body consisting of a plastic material

[4] E. Melan, Ing. Arch. **9**, 116–126 (1938).

[5] See for instance W. Lode, Forschungsarbeiten a.d. Gebiete Ingenieurwesens No. 303, VDI-Verlag, Berlin 1928; G. I. Taylor and H. Quinney, Phil. Trans. Roy. Soc. London (A) **230**, 323–362 (1931); W. Prager, J. Appl. Phys. **16**, 837–840 (1945).

[6] Since we are only interested in the sign of df, we could always replace df by $d\varphi(f)$ where $\varphi'(f)$ was positive definite. Thus the fact that the expression (10) could be denoted by $d\varphi(f)$, instead of just df, does not really add any generality.

which obeys the stress-strain relation (15). Given the state of stress throughout this body (initial condition) and the rates of loading on its surface (boundary condition), we wish to determine the stress rates throughout the body.

Let $\dot{\sigma}_{ij}$ denote the stress rates which actually occur under these conditions ("actual" stress rates) and $\dot{\sigma}'_{ij}$ any set of "artificial" stress rates which satisfy equilibrium and boundary conditions. Furthermore, let $\dot{\epsilon}_{ij}$ and $\dot{\epsilon}'_{ij}$ denote the corresponding strain rates, determined in accordance with Eq. (15) from the initial stress σ_{ij} and the stress rates $\dot{\sigma}_{ij}$ and $\dot{\sigma}'_{ij}$, respectively. The artificial strain rates $\dot{\epsilon}'_{ij}$ will not satisfy the compatibility equations as a rule.

We shall now prove the following lemmas.

LEMMA 1. *For the actual stress rate $\dot{\sigma}_{ij}$, the artificial stress rate $\dot{\sigma}'_{ij}$, and the corresponding rates of strain,*

$$(\dot{\sigma}_{ij} - \dot{\sigma}'_{ij})(\dot{\epsilon}_{ij} - \dot{\epsilon}'_{ij}) > 0 \tag{16}$$

PROOF: The substitution of (15) into the left-hand side of (16) yields

$$(\dot{\sigma}_{ij} - \dot{\sigma}'_{ij})(\dot{\epsilon}_{ij} - \dot{\epsilon}'_{ij}) = (\dot{\sigma}_{ij} - \dot{\sigma}'_{ij})(\dot{\epsilon}^{(0)}_{ij} - \dot{\epsilon}^{(0)\prime}_{ij}) + (\dot{\sigma}_{ij} - \dot{\sigma}'_{ij})\tfrac{1}{2}F(\partial f/\partial\sigma_{ij})[(f + |f|) - (f' + |f'|)]. \tag{17}$$

If we substitute (2) and (14) into the first term on the right-hand side of (17) we obtain

$$(\dot{\sigma}_{ij} - \dot{\sigma}'_{ij})(\dot{\epsilon}^{(0)}_{ij} - \dot{\epsilon}^{(0)\prime}_{ij}) = (1 + \nu)(\dot{s}_{ij} - \dot{s}'_{ij})(\dot{s}_{ij} - \dot{s}'_{ij}) + 3(1 - 2\nu)(\dot{s} - \dot{s}')^2,$$

which is positive, since[7]

$$-1 \leq \nu \leq 1/2. \tag{18}$$

To determine the sign of the other term several cases must be considered.

(*i*) If both the actual and artificial states constitute loading, then in view of the relations

$$f = (\partial f/\partial\sigma_{pq})\dot{\sigma}_{pq}, \qquad f' = (\partial f/\partial\sigma_{pq})\dot{\sigma}'_{pq}, \tag{19}$$

the last term on the right-hand side of (17) becomes

$$(\dot{\sigma}_{ij} - \dot{\sigma}'_{ij})F(\partial f/\partial\sigma_{ij})(f - f') = F(f - f')^2,$$

which is non-negative, since F is positive definite.

(*ii*) If the actual state constitutes loading and the artificial state unloading, then the last term on the right-hand side of (17) becomes

$$(\dot{\sigma}_{ij} - \dot{\sigma}'_{ij})F(\partial f/\partial\sigma_{ij})f = F(f - f')f,$$

which is positive, since in this case

$$f > 0, \qquad f' < 0.$$

[7] See for instance A. E. H. Love, *A treatise on the mathematical theory of elasticity*, 4th edition, Dover Publications, New York, 1946, p. 104.

(*iii*) If the actual state constitutes unloading and the artificial state loading, then the last term on the right-hand side of (17) becomes

$$(\dot\sigma_{ij} - \dot\sigma'_{ij})F(\partial f/\partial\sigma_{ij})(-f') = F(f - f')(-f'),$$

which is positive, since in this case

$$f < 0, \qquad f' > 0.$$

(*iv*) Finally, if both the actual and artificial states constitute unloading, the last term on the right-hand side of (17) is identically equal to zero.

COROLLARY. *For any stress rate* $\dot\sigma_{ij}$ *and the corresponding strain rate* $\dot\epsilon_{ij}$,

$$\dot\sigma_{ij}\dot\epsilon_{ij} > 0. \tag{20}$$

The proof is precisely analogous to that of the lemma.

LEMMA 2. *For the natural stress rate* $\dot\sigma_{ij}$, *the artificial stress* $\dot\sigma'_{ij}$, *and the corresponding rates of elastic strain,*

$$\dot\sigma_{ij}\dot\epsilon^{(0)\prime}_{ij} = \dot\sigma'_{ij}\dot\epsilon^{(0)}_{ij}. \tag{21}$$

This is proved by substituting (14) in both sides of (21) and recalling definitions (1) and (2).

LEMMA 3. *If both the actual and artificial stress rates constitute loading, then*

$$\dot\sigma_{ij}\dot\epsilon'_{ij} = \dot\sigma'_{ij}\dot\epsilon_{ij}. \tag{22}$$

This is proved by substituting (15) in both sides of (22) and using (21) and (19).

LEMMA 4. (*An analogy to the principle of virtual displacements.*) Consider any system of stress rates $\dot\sigma_{ij}$ and a system of strain rates $\dot\epsilon^*_{ij}$ which are defined in a region V and satisfy the following conditions:

(*i*) the stress rates satisfy the equations of equilibrium

$$\dot\sigma_{ij,i} = 0$$

and the boundary condition

$$\dot T_j = \dot\sigma_{ij}n_i,$$

where the rates $\dot T_j$ are given over the entire surface S of V;

(*ii*) the strain rates satisfy the equations of compatibility, i.e. they are derived from velocities $\dot u^*_i$ in accordance with

$$\dot\epsilon^*_{ij} = \tfrac{1}{2}(\dot u^*_{i,j} + \dot u^*_{j,i});$$

(*iii*) the region V can be divided into a finite number of sub-regions $V^{(1)}, V^{(2)}, \cdots, V^{(n)}$ such that $\dot\sigma_{ij}$ and $\dot\epsilon^*_{ij}$ are analytic in each of these sub-regions;

(*iv*) The velocities $\dot u^*_i$ are continuous throughout V and;

(*v*) the rates of the stresses transmitted across the surface of separation between the regions $V^{(p)}$ and $V^{(q)}$ are the same whether they are evaluated from the expression of $\dot\sigma_{ij}$ valid in $V^{(p)}$ or from that valid in $V^{(q)}$.

It is not necessary that $\dot\sigma_{ij}$ and $\dot\epsilon^*_{ij}$ be connected by the stress-strain law.

Under these five conditions it can be shown by precisely the method used to prove the principle of virtual displacements, that

$$\int \dot{\sigma}_{ij}\dot{\epsilon}^*_{ij}\,dv = \int \dot{T}_i\dot{u}^*_i dS. \tag{23}$$

4. The stress-strain relation (15) satisfies the conditions formulated in Section 1. To show that the irreversibility condition (9) is fulfilled, we substitute (11) into the left-hand side of (9). Thus,

$$2pJ_2 + 3qJ_3^2 = F[2(\partial f/\partial J_2)J_2 + 6(\partial f/\partial J_3^2)J_3^2]. \tag{24}$$

Since J_2 is a *quadratic* form and J_3 a *cubic* form in the components of the stress deviation and since we demand that $f(J_2, J_3^2)$, too, be a homogeneous in these components, we may use Euler's theorem to write (24) in the form

$$2pJ_2 + 3qJ_3^2 = nFf,$$

where n is the degree of f. Therefore, since F and f are positive definite, the condition of irreversibility (9) is satisfied by our choice of f.

If the state of stress σ_{ij} is given throughout the body and the stress rates $\dot{T}_j = \dot{\sigma}_{ij}n_i$ on the entire surface, the stress rates throughout the body will be uniquely determined. To prove this proposition, we start from the assumption that it is not correct. Let $\dot{\sigma}^{(a)}_{ij}$ and $\dot{\sigma}^{(b)}_{ij}$ be two sets of stress rates which satisfy the equations of equilibrium throughout the body, and the boundary conditions

$$\dot{T}_j = \dot{\sigma}^{(a)}_{ij}n_i = \dot{\sigma}^{(b)}_{ij}n_i$$

on the entire surface. Furthermore, let $\dot{\epsilon}^{(a)}_{ij}$ and $\dot{\epsilon}^{(b)}_{ij}$ be the strain rates determined from $\dot{\sigma}^{(a)}_{ij}$ and $\dot{\sigma}^{(b)}_{ij}$ by means of Eq. (15). Since we assume both sets of stress and strain rates to constitute solutions of our boundary value problem, $\dot{\epsilon}^{(a)}_{ij}$ as well as $\dot{\epsilon}^{(b)}_{ij}$ will satisfy the equations of compatibility. Thus the strain rates $\dot{\epsilon}^{(a)}_{ij}$ are derived from velocities $\dot{u}^{(a)}_i$, and the strain rates $\dot{\epsilon}^{(b)}_{ij}$ from velocities $\dot{u}^{(b)}_i$.

If $\dot{\sigma}^{(a)}_{ij}$ is not identically equal to $\dot{\sigma}^{(b)}_{ij}$, then we may integrate the inequality of Lemma 1 over the entire volume and obtain,

$$\int (\dot{\sigma}^{(a)}_{ij} - \dot{\sigma}^{(b)}_{ij})(\dot{\epsilon}^{(a)}_{ij} - \dot{\epsilon}^{(b)}_{ij})dv > 0. \tag{25}$$

If we use the notations

$$\dot{\sigma}^{(c)}_{ij} = \dot{\sigma}^{(a)}_{ij} - \dot{\sigma}^{(b)}_{ij},\ \dot{\epsilon}^{(c)}_{ij} = \dot{\epsilon}^{(a)}_{ij} - \dot{\epsilon}^{(b)}_{ij},\ \dot{u}^{(c)}_i = \dot{u}^{(a)}_i - \dot{u}^{(b)}_i,$$

then $\dot{\sigma}^{(c)}_{ij}$ will correspond to the rates of loading

$$\dot{T}^{(c)}_j = \dot{\sigma}^{(c)}_{ij}n_i = 0 \tag{26}$$

on the entire surface. Also, since $\dot{\sigma}^{(c)}_{ij}$ and $\dot{\epsilon}^{(c)}_{ij}$ satisfy the five conditions of Lemma 4,

$$\int \dot{\sigma}^{(c)}_{ij}\dot{\epsilon}^{(c)}_{ij}dv = \int \dot{T}^{(c)}_i\dot{u}^{(c)}_i dS. \tag{27}$$

But in view of (26), the right-hand side of (27) equals zero, and hence (27) contradicts (25). Therefore $\dot{\sigma}_{ij}^{(a)} \equiv \dot{\sigma}_{ij}^{(b)}$ and the stress rates are uniquely determined by the rates of loading on the surface.

5. Minimum principle for loading throughout the body. In this section we shall assume that both the actual and artificial stress rates constitute loading throughout the body. With this limitation it will be shown that the volume integral of the invariant $\dot{\sigma}_{ij}\dot{e}_{ij}$ over the entire body has a smaller value for the actual stress rates than for any artificial system of stress rates.

Let us first take the integral of (16) over the body. We obtain

$$A - 2B + D > 0, \tag{28}$$

where

$$A = \int \dot{\sigma}_{ij}\dot{e}_{ij}\, dv, \tag{29-a}$$

$$B = \frac{1}{2}\int (\dot{\sigma}'_{ij}\dot{e}_{ij} + \dot{\sigma}_{ij}\dot{e}'_{ij})\, dv, \tag{29-b}$$

$$D = \int \dot{\sigma}'_{ij}\dot{e}'_{ij}\, dv. \tag{29-c}$$

From Lemma 3, it follows that

$$B = \int \dot{\sigma}'_{ij}\dot{e}_{ij}\, dv. \tag{30}$$

Since $\dot{\sigma}'_{ij}$ and $\dot{e}_{ij}$ satisfy the five conditions of Lemma 4, it follows from (23) that

$$B = \int \dot{\sigma}'_{ij}\dot{e}_{ij}\, dv = \int \dot{T}'_i \dot{u}_i\, dS, \tag{31}$$

where the integral is to be taken over the entire surface of the body. However, since both the actual and artificial stress rates must satisfy the same boundary conditions everywhere on the surface, we have $\dot{T}'_i = \dot{T}_i$ and hence

$$B = \int \dot{T}_i \dot{u}_i\, dS.$$

Since $\dot{\sigma}_{ij}$ also satisfies the conditions of Lemma 4, we can again apply (23) to show that

$$B = \int \dot{\sigma}_{ij}\dot{e}_{ij}\, dv = A. \tag{32}$$

Substituting (32) into (28) we obtain $A < D$, or

$$\int \dot{\sigma}_{ij}\dot{e}_{ij}\, dv < \int \dot{\sigma}'_{ij}\dot{e}'_{ij}\, dv. \tag{33}$$

This establishes the minimum principle in the case where both the natural and the artificial state constitute loading throughout the body.

6. Extension to other cases. Consider a body under a system of surface forces such that both the actual and artificial stress rates constitute loading in some parts and unloading in others. Let V_{11} denote that part of the body which undergoes loading under both the actual and artificial stress rates, V_{12} that part which is loaded according to the actual rates, but unloaded according to the artificial stress rates, V_{21} that part which is being unloaded according to the actual rates, but loaded according to the artificial rates and, finally V_{22} that part which undergoes unloading under both the actual and the artificial stress rates. If we particularize Eq. (15) for both the actual and artificial stress rates, and for all four regions, we obtain the following stress-strain relations:

$$\text{in } V_{11}\text{: } \dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^{(0)} + F(\partial f/\partial\sigma_{ij})\dot{f}, \qquad \dot{\epsilon}'_{ij} = \dot{\epsilon}_{ij}^{(0)'} + F(\partial f/\partial\sigma_{ij})\dot{f}'; \tag{34-a}$$

$$\text{in } V_{12}\text{: } \dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^{(0)} + F(\partial f/\partial\sigma_{ij})\dot{f}, \qquad \dot{\epsilon}'_{ij} = \dot{\epsilon}_{ij}^{(0)'}; \tag{34-b}$$

$$\text{in } V_{21}\text{: } \dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^{(0)}, \qquad \dot{\epsilon}'_{ij} = \dot{\epsilon}_{ij}^{(0)'} + F(\partial f/\partial\sigma_{ij})\dot{f}'; \tag{34-c}$$

$$\text{in } V_{22}\text{: } \dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^{(0)}, \qquad \dot{\epsilon}'_{ij} = \dot{\epsilon}_{ij}^{(0)'}. \tag{34-d}$$

Since the equations proved in Lemmas 2 and 3 are local equations, they are valid wherever the required conditions are fulfilled. Applying Lemma 3 to V_{11}, and Lemma 2 to the remaining regions, we obtain the following equalities:

$$\text{in } V_{11}\text{: } \dot{\sigma}_{ij}\dot{\epsilon}'_{ij} = \dot{\sigma}'_{ij}\dot{\epsilon}_{ij}; \tag{35-a}$$

$$\text{in } V_{12}\text{: } \dot{\sigma}_{ij}\dot{\epsilon}'_{ij} = \dot{\sigma}_{ij}\dot{\epsilon}_{ij}^{(0)'} = \dot{\sigma}'_{ij}\dot{\epsilon}_{ij}^{(0)} = \dot{\sigma}'_{ij}\dot{\epsilon}_{ij} - F\dot{f}\dot{f}'; \tag{35-b}$$

and similarly,

$$\text{in } V_{21}\text{: } \dot{\sigma}_{ij}\dot{\epsilon}'_{ij} = \dot{\sigma}'_{ij}\dot{\epsilon}_{ij} + F\dot{f}\dot{f}'; \tag{35-c}$$

$$\text{and in } V_{22}\text{: } \dot{\sigma}_{ij}\dot{\epsilon}'_{ij} = \dot{\sigma}'_{ij}\dot{\epsilon}_{ij}. \tag{35-d}$$

Applying equations (35) to (29-b), we obtain,

$$B = \int_V \dot{\sigma}'_{ij}\dot{\epsilon}_{ij}\,dv - \frac{1}{2}\int_{V_{12}} F\dot{f}\dot{f}'\,dv + \frac{1}{2}\int_{V_{21}} F\dot{f}\dot{f}'\,dv, \tag{36}$$

where

$$V = V_{11} + V_{12} + V_{21} + V_{22}$$

is the total volume.

The quantities appearing in the first integral on the right-hand side of (36) satisfy the five conditions of Lemma 4, and therefore the integral may be transformed in exactly the same manner as in Sec. 5 to obtain

$$B = \int_V \dot{\sigma}_{ij}\dot{\epsilon}_{ij}\,dv + H = A + H, \tag{37}$$

where

$$H = -\frac{1}{2}\int_{V_{12}} F\dot{f}\dot{f}'\,dv + \frac{1}{2}\int_{V_{21}} F\dot{f}\dot{f}'\,dv. \tag{38}$$

In view of (37), the inequality (28) can now be written as

$$D > A + 2H. \tag{39}$$

Therefore, if we substitute (29) and (38) into (39), we obtain the inequality

$$\int_V \dot{\sigma}_{ij}\dot{e}_{ij}\,dv < \int_V \dot{\sigma}'_{ij}\dot{e}'_{ij}\,dv + \int_{V_{12}} F\dot{f}\dot{f}'\,dv - \int_{V_{21}} F\dot{f}\dot{f}'\,dv. \tag{40}$$

Although (40) holds in all cases, it does not lend itself to an application of the direct methods of variational calculus, since the last two integrals depend on both the actual and the artificial stress rates. However, if these stress rates are such that $H \geq 0$, then (40) will reduce to the inequality (33) derived for loading only. There are two practically important situations where this will be the case.

Since $\dot{f} > 0$ for loading and $\dot{f} < 0$ for unloading, and since F is positive definite,

$$F\dot{f}\dot{f}' < 0 \tag{41}$$

throughout both V_{12} and V_{21}. Thus a sufficient condition for H to be non-negative is that $V_{21} = 0$. Therefore, we may conclude that (33) will hold in all cases where the artificially loaded region is entirely contained in the actually loaded region.

If we admit only those artificial stress rates which are indefinitely close to the actual stress rates in the neighborhood of the dividing surface between loading and unloading, then it can be shown that H is an infinitesimal of higher order than A and D.[8] Therefore, in this case also the minimum principle (33) will apply.

7. Conclusion and limitations. Our conclusion, then, is as follows: *The volume integral of the invariant $\dot{\sigma}_{ij}\dot{e}_{ij}$ over the entire body is less for the actual stress rates than it is for any system of artificial stress rates which satisfy equilibrium and boundary conditions, and which either are indefinitely near the actual stress rates, or constitute unloading in all regions of the body where the actual stress rates constitute unloading.*

A similar procedure may be used to derive a minimum principle for the mixed boundary value problem, where the stresses are given throughout the interior of the body, the stress rates on a part of the surface, S_B, and the velocities on the remainder of the surface S_A. Let

$$\Pi^* = \frac{1}{2}\int_V \dot{\sigma}_{ij}\dot{e}_{ij}\,dv - \int_{S_A} \dot{T}_i\dot{u}_i\,dS_A. \tag{42}$$

Then it can be shown that Π^* is less for the actual stress rates than for any system of artificial stress rates which satisfy the above restrictions. In the case where the velocities are prescribed zero on S_A, this principle will reduce to (33).

These conclusions are, of course, subject also to the limitations imposed on the stress-strain relation developed by Handelman, Lin, and Prager. Therefore, it

[8] In this connection, see W. Prager, *Variational Principles in the Theory of Plasticity*, to be published in Proc. 6th Internat. Congr. Appl. Mech., Paris, 1946, where the problem is treated by methods of the calculus of variations.

applies only to cases where loading from the stress-free natural state is followed by at most *one* unloading, and only to materials which exhibit strain-hardening.

It is also worth while pointing out that in the case where the artificially loaded region is entirely contained within the naturally loaded region, we have proved an *absolute* minimum principle, since we have proved actual inequalities. Thus our conclusion is stronger than if we had used the method of first and second variations which can never do more than compare the natural state with indefinitely near artificial states.

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AUXILIARY TABLE FOR THE INCOMPLETE ELLIPTIC INTEGRALS

By E. L. Kaplan

1. Introduction. Interpolation into Legendre's tables [1a or b] of the incomplete elliptic integrals F and E is not practicable when both the modular angle $\sin^{-1}k$ and the amplitude φ are near 90°. The present paper derives and tabulates easily interpolable functions from which the integrals may be calculated in this case. The relevant formulas are (1) and (2) below. A new list of errata for Legendre's tables is given.

2. Derivation of the formulas.

$$K - F(k,\varphi) = \int_{\sin\varphi}^{1} \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}} = \int_0^{\cos\varphi} \frac{dt}{\sqrt{(1-t^2)(k'^2+k^2t^2)}}$$

$$= \int_0^x \frac{dt}{k\sqrt{r^2+t^2}}\left(1 + \frac{1}{2}t^2 + \frac{1\cdot3}{2\cdot4}t^4 + \cdots\right), \quad \text{where } x = \cos\varphi,$$

$r = k'/k$. We use the formulas

$$\int_0^x \frac{dt}{\sqrt{r^2+t^2}} = \ln\frac{x+\sqrt{r^2+x^2}}{r} = \sinh^{-1}\frac{x}{r}$$

$$\int_0^x \frac{t^m\,dt}{\sqrt{r^2+t^2}} = \frac{x^{m-1}}{m}\sqrt{r^2+x^2} - r^2\frac{m-1}{m}\int_0^x \frac{t^{m-2}\,dt}{\sqrt{r^2+t^2}}$$

$$\therefore K - F(k,\varphi) = \left[1 - \frac{1}{2}\cdot\frac{r^2}{2} + \frac{1\cdot3}{2\cdot4}\cdot\frac{3}{4}\cdot\frac{1}{2}r^4 - \cdots\right]\frac{1}{k}\sinh^{-1}\frac{x}{r}$$

$$+ \frac{1}{k}\sqrt{r^2+x^2}\left[\frac{1}{2}\cdot\frac{x}{2} + \frac{1\cdot3}{2\cdot4}\left(\frac{x^3}{4} - \frac{3}{4}r^2\cdot\frac{x}{2}\right)\right.$$

$$\left. + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(\frac{x^5}{6} - \frac{5}{6}r^2\cdot\frac{x^3}{4} + \frac{5\cdot3}{6\cdot4}r^4\cdot\frac{x}{2}\right) + \cdots\right]$$

The expression multiplying $\sinh^{-1}x/r$ is the expansion of

$$\frac{2}{\pi k}\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1+r^2\sin^2\varphi}} = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k'^2\cos^2\varphi}} = \frac{2}{\pi}K(k').$$

Hence

$$K - F(k,\varphi) = \frac{2K'}{\pi}\sinh^{-1}\frac{x}{r} + x\sqrt{r^2+x^2}\cdot f \tag{1}$$

where

$$f/\sqrt{1+r^2} = \left[\left(\frac{1}{2}\right)^2 - \left(\frac{1\cdot3}{2\cdot4}\right)^2 r^2 + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 r^4 - \cdots\right]$$

$$+ \frac{2}{3}x^2\left[\left(\frac{1\cdot3}{2\cdot4}\right)^2 - \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 r^2 + \cdots\right]$$

$$+ \frac{2\cdot4}{3\cdot5}x^4\left[\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 - \left(\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\right)^2 r^2 + \cdots\right] + \cdots$$

For the integral of the second kind we have

$$E - E(k,\varphi) = \int_{\sin\varphi}^{1} \sqrt{\frac{1-k^2s^2}{1-s^2}}\,ds = \int_0^{\cos\varphi} \sqrt{\frac{k'^2+k^2t^2}{1-t^2}}\,dt$$

$$= \int_0^x k\sqrt{r^2+t^2}\,dt\left(1+\frac{1}{2}t^2+\frac{1\cdot3}{2\cdot4}t^4+\cdots\right).$$

We use the formulas

$$\int_0^x \sqrt{r^2+t^2}\,dt = \frac{x}{2}\sqrt{r^2+x^2}+\frac{r^2}{2}\sinh^{-1}\frac{x}{r}$$

$$\cdot\int_0^x t^m\sqrt{r^2+t^2}\,dt = \frac{x^{m-1}}{m+2}(r^2+t^2)^{3/2} - r^2\frac{m-1}{m+2}\int_0^x t^{m-2}\sqrt{r^2+t^2}\,dt.$$

$$\therefore E - E(k,\varphi) = \left[1-\frac{1}{2}\cdot\frac{r^2}{4}+\frac{1\cdot3}{2\cdot4}\cdot\frac{3}{6}\cdot\frac{1}{4}r^4-\cdots\right]k\int_0^x\sqrt{r^2+t^2}\,dt$$

$$+k(r^2+x^2)^{3/2}\left[\frac{1}{2}\cdot\frac{x}{4}+\frac{1\cdot3}{2\cdot4}\left(\frac{x^2}{6}-\frac{3}{6}r^2\cdot\frac{x}{4}\right)\right.$$

$$\left.+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(\frac{x^5}{8}-\frac{5}{8}r^2\cdot\frac{x^3}{6}+\frac{5\cdot3}{8\cdot6}r^4\cdot\frac{x}{4}\right)+\cdots\right].$$

The expression multiplying the integral is the expansion of

$$\frac{2k}{\pi}\int_0^{\pi/2}\frac{2\cos^2\varphi\,d\varphi}{\sqrt{1+r^2\sin^2\varphi}} = \frac{4k^2}{\pi}\int_0^{\pi/2}\frac{\cos^2\varphi\,d\varphi}{\sqrt{1-k'^2\cos^2\varphi}} = \frac{4}{\pi r^2}[K(k')-E(k')].$$

Hence

$$E - E(k,\varphi) = \frac{2}{\pi}(K'-E')\left[\sinh^{-1}\frac{x}{r}+\frac{x}{r^2}\sqrt{r^2+x^2}\right]+x(r^2+x^2)^{3/2}\cdot e^*,$$

where

$$e^*\sqrt{1+r^2} = \frac{2}{1}\left[\left(\frac{1}{2}\right)^2\frac{1}{4}-\left(\frac{1\cdot3}{2\cdot4}\right)^2\cdot\frac{r^2}{6}+\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2\cdot\frac{r^4}{8}-\cdots\right]$$

$$+\frac{2\cdot4}{1\cdot3}x^2\left[\left(\frac{1\cdot3}{2\cdot4}\right)^2\cdot\frac{1}{6}-\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2\cdot\frac{r^2}{8}+\cdots\right]+\cdots$$

If we set $e = 2(K'-E')/\pi r^2 + (r^2+x^2)e^*$, we have

$$E - E(k,\varphi) = \frac{2}{\pi}(K'-E')\sinh^{-1}\frac{x}{r}+x\sqrt{r^2+x^2}\cdot e, \tag{2}$$

an equation which differs from (1) in the right member only in having $K'-E'$ and e in place of K' and f, respectively. The equation for e can be reduced to

$$e\cdot\sqrt{1+r^2} = (\sin^{-1}x)/x - \left[\frac{1}{2} - \left(\frac{1}{2}\right)^2\cdot\frac{r^2}{4} + \left(\frac{1\cdot3}{2\cdot4}\right)^2\cdot\frac{r^4}{6}\cdots\right]$$
$$- \frac{2}{3}x^2\left[\left(\frac{1}{2}\right)^2\cdot\frac{1}{4} - \left(\frac{1\cdot3}{2\cdot4}\right)^2\cdot\frac{r^2}{6} + \cdots\right]$$
$$- \frac{2\cdot4}{3\cdot5}x^4\left[\left(\frac{1\cdot3}{2\cdot4}\right)^2\cdot\frac{1}{6} - \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2\cdot\frac{r^2}{8} + \cdots\right] - \cdots.$$

Since this formula was not discovered until after e^* had been computed, the values of e were computed from 10-decimal values of e^* rather than directly.

It is possible to modify equations (1) and (2) to give $F(k, \varphi)$ and $E(k, \varphi)$ directly, K and E not appearing:

$$F(k,\varphi) = \frac{2K'}{\pi}\ln\frac{4}{k}(x + \sqrt{r^2+x^2})^{-1} - x\sqrt{r^2+x^2}\cdot f - K_2$$

$$E(k,\varphi) = \frac{2}{\pi}(K' - E')\ln\frac{4}{k}(x + \sqrt{r^2+x^2})^{-1} - x\sqrt{r^2+x^2}\cdot e + E_2$$

(3)

$$K_2 = \left(\frac{1}{2}\right)^2\cdot\frac{2}{1\cdot2}k'^2 + \left(\frac{1\cdot3}{2\cdot4}\right)^2\left(\frac{2}{1\cdot2} + \frac{2}{3\cdot4}\right)k'^4 + \cdots$$

$$E_2 = 1 - \frac{1}{2}\cdot\frac{1}{1\cdot2}k'^2 - \left(\frac{1}{2}\right)^2\cdot\frac{3}{4}\cdot\left(\frac{2}{1\cdot2} + \frac{1}{3\cdot4}\right)k'^4$$
$$- \left(\frac{1\cdot3}{2\cdot4}\right)^2\cdot\frac{5}{6}\cdot\left(\frac{2}{1\cdot2} + \frac{2}{3\cdot4} + \frac{1}{5\cdot6}\right)k'^6 - \cdots$$

K_2 and E_2 have been tabulated by Airey [2]. Equations (3) are a little more general than (1) and (2), since (3) could be used to compute the complete integrals if desired. A previous paper [3] by the author provides a more direct means for doing this, however, so that if it is available the more complicated equations (3) would be useful only in the uncommon case that the complete integrals were unknown and unwanted.

3. Description of the tables. The principal table gives 10-decimal values of f and e for $x^2 = -.005(.005)0.16$, $r^2 = -.005(.005)0.16$, the negative arguments being included to facilitate interpolation. Fourth differences are everywhere less than about 16 units in the last place, and may be neglected. The greatest second and third differences are 7×10^{-6} and 8×10^{-8}, so that linear and quadratic interpolation is substantially valid to 6 and 8 decimals, respectively. The table is so constructed that even if the initial data are values of x^2 and k^2, only one square root has to be extracted, namely $\sqrt{x^2(r^2+x^2)}$. For this purpose the expression $\sinh^{-1} x/r$ is written in the form

$$\tfrac{1}{2}\sinh^{-1}(2x\sqrt{r^2+x^2}/r^2) = \tfrac{1}{2}\ln[(r^2+2x^2+2x\sqrt{r^2+x^2})/r^2].$$

All of the values were calculated directly from the double series, and the results checked by differencing. Tabular errors should not exceed 1×10^{-10}. As a

check on the formulas and general procedure, values of both integrals for $x^2 = 0.09$, $r^2 = 0.16$ were calculated by numerical integration; the results agreed with (1) and (2) to 12 decimals.

As a by-product of the principal computations, 12-decimal values of the functions $2K'/\pi$ and $2(K' - E')/\pi$ appearing in (1) and (2) were obtained as functions of r^2 for $r^2 = -.005(.005)0.16$ and are included here. The last decimal of $2K'/\pi$ is not precise; and the fourth differences of $2(K' - E')/\pi$, which vary from -3000 to -1600×10^{-12}, are not perfectly allowed for by the modified second differences, so that a maximum error of 1.3×10^{-12} may arise thereby.

The author's previous table [3] gives the complete elliptic integrals K and E near their singularity as functions of $\log_{10} k'^2$. A similar device is possible for the incomplete integrals, by using as arguments $\lambda = \sinh^{-1} x/r$, $\alpha = r^2 + x^2$. The inverse relations are $r^2 = \alpha \operatorname{sech}^2 \lambda$, $x^2 = \alpha \tanh^2 \lambda$, $x\sqrt{r^2 + x^2} = \alpha \tanh \lambda$. The fact that such a treatment increases the size of the required table is of little consequence for the univariate table, but it would be a grave disadvantage for the bivariate table. The labor saved by the novel form of table would be about the same in each case, taken absolutely, and hence of little importance, relatively, for the incomplete integrals.

4. Checking of Legendre's Tables. There is one other respect in which Legendre's tables [1] need to be supplemented, and that is by a list of errata. In an effort to find most of the errors detectable by a reasonable amount of work, the following operations were performed on the Pearson edition: (a) In Legendre's Table I the indicated differences of $\log_{10} E'$ and $\log_{10} F'$ were verified (this disregards the 13th and 14th decimals given in part of the table). (b) In Legendre's Table IX all those fifth differences of E and F with respect to θ which involve entries on the same printed page (i.e., every fifth such difference) were calculated. (c) For the columns in which θ is a multiple of 5°, and the preceding test insensitive, every $(n - 1)$st one of the nth differences ($n = 4$ or 6) with respect to φ was calculated. (d) Values occurring at more than one place in Legendre's nine tables were compared with one another as well as with anti-logarithms calculated from Table I at intervals of 1°, and complete integrals calculated from Hayashi [5] and Kaplan [3] for $\theta' = 0(5°)80°(1°)90°$. (e) Entries in Table IX for $\theta = 0$ and 90° were compared with other tables. (f) As a spot-check in the difficult region, E and F were computed from the present tables for θ and $\varphi = 70°(5°)85°$. The results are given below. The largest error in the set is 2.4×10^{-9}. (g) Heuman's values [4] of $F(86°, 86°)$ and $F(87°, 87°)$were corrected in the ninth decimal, and F was computed by numerical integration and checked by the present tables for $\theta = 76°, \varphi = 55(1)89°$, where Legendre has systematic errors $\leq 31 \times 10^{-9}$, and one other.

In the following table of errata, those previously given by Heuman [4] and/or Samoilova-Yakhontova [6] are starred. As they had already been entered into the copy being checked, they were not subject to rediscovery by the writer. The letter D indicates that corresponding errors occur in higher differences. The letter N indicates that the error does not occur in the 1816 edition and its reprints, which were only examined for the errors found in the Pearson edition.

The obscurities noted, some of which led to errors in checking, are all peculiar to the Pearson edition, except for one in log E' (22.4°) in the 1816 edition, which resulted in an error in the 1826. Page numbers in parentheses refer to the 1826 edition.

For p. 86 (355), function F, $\theta = 76°$, $\varphi = 55(1)\ 89°$, the amounts to be added ($\times 10^{-9}$) to Legendre's values, and the resulting correct values for the last three digits (7–9th decimals) are as follows:

φ = 55–60:					21	−179	21	21	21	21
					100	376	946	923	475	964
φ = 61–70:	22	22	23	23	24	25	26	26	27	27
	088	024	575	302	656	083	101	331	473	176
φ = 71–80:	28	29	30	30	31	31	29	27	25	23
	809	051	284	703	082	095	120	398	488	950
φ = 81–89:	21	19	16	15	13	11	9	6	3	
	265	111	247	485	391	625	843	066	042	

Also, for $\varphi = 45°$ in Table IX, decrease by one unit in the last place $E(24°)$, $E(60°)$, $F(89°)$; increase E by one unit for $\theta = 38, 53, 62, 64, 70, 74, 78$, and F for $\theta = 1, 6, 20, 46, 60, 65, 75$. (This is based on Legendre's own 12-decimal values in Table VIII.)

The table of errors does not include 14 errors of 1 or 2 units, presumably only typical of considerable portions of Table IX, derived from the following values calculated by the writer:

	φ	θ = 70°	75°	80°	85°
E:	70°	.98297 58254	.96518 62639	.95143 84692	.94269 81311
	75°	1.02171 63337	.99915 74432	.98140 78139	.96992 21180
	80°	1.05648 22128	1.02823 30506	1.00543 29463	.99022 77900
	85°	1.08824 77295	1.05342 63162	1.02436 39317	1.00394 02679
F:	φ	θ = 70°	75°	80°	85°
	70°	1.59590 62431	1.64683 71117	1.69181 48925	1.72372 39526
	75°	1.79268 73574	1.87145 39624	1.94682 23053	2.00498 77555
	80°	2.01192 79796	2.13389 51423	2.26527 32608	2.38364 70898
	85°	2.25177 99498	2.43657 61369	2.66935 04480	2.94868 87547

The consistent discrepancy of one unit between the two sets of given values of F for $\theta = 75°$, $\varphi = 84$–$88°$ is curious. Only the value for $\varphi = 85°$ was recalculated. Another curious fact concerns the errors in E for $\varphi = 7°$, $\theta = 11°$, and $\varphi = 6°$, $\theta = 86°$; and in F for $\varphi = 6°$, $\theta = 14°$, and $\varphi = 31°$, $\theta = 83°$. These were not given by Heuman, although he states that he used the 1826 edition (none of them occurs in the 1816), and if uncorrected they would have resulted in errors in his 6-decimal tables (his values are actually correct).

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REFERENCES

[1a] LEGENDRE, A. M.: Exercices de Calcul intégral . . . , Paris, 1816, Tome 3. Table IX reprinted in L. Potin, Formules et tables numériques, Paris 1925. Tables VIII and IX reprinted by Fritz Emde, Stuttgart 1931.

[1b] LEGENDRE, A. M.: Traité des fonctions elliptiques, Paris, 1826. Tome 2, pp. 222–243; 232–363.
Tables I and IX reprinted in 1934 by Karl Pearson, Biometrika Office, London.

[2] AIREY, J. R.: Toroidal Functions and the Complete Elliptic Integrals. Phil. Mag., Ser. 7, Vol. 19, 1935, pp. 177–188.

[3] KAPLAN, E. L.: Auxiliary Table of Complete Elliptic Integrals. J. Math. Phys. Vol. 25, 1946, pp. 26–36.

[4] HEUMAN, CARL: Tables of Complete Elliptic Integrals. J. Math. Phys. Vol. 22, 1941, pp. 127–206.

[5] HAYASHI, K.: Tafeln der Besselschen, Theta-, Kugel-, und anderer Funktionen (Table of K, pp. 72–81). Springer (Berlin), 1930. Tafeln für die Differenzrechnung, etc. (Table of E, pp. 62–65). Springer (Berlin), 1933.

[6] SAMOILOVA-YAKHONTOVA: Tables of Elliptic Integrals. Astronomical Institute of Leningrad, 1935.

ERRATA FOR TABLE I

PAGE	θ	COLUMN	FOR	READ	
1(222)	1.2	F^1 I	... 275	8 269 175	
3(224)	14.9	E^1 II	... 932	654 935	D
4(225)	15.5	E^1 III	106	104	
"	19.2	E^1 III	151	149	
5(226)	22.0	E^1 III	188	192	
"	22.4	E^1	... 460	0.179 585 059 469	obsc.
"	22.9	F^1 II	... 724	734 727	N
"	24.9	F^1	..287 732	0.217 047 387 732*	N
6(227)	28.7	E^1 III	332	334	
6—	29.7	E^1	obscure	0.167 149 995 573	N
7 228.	30.9	F^1	8.228 ...	0.228 724 167 129 *	N
"	34.4	E^1	..108 854	0.157 376 100 854 *	
8(229)	36.2	E^1	... 718	0.153 276 754 518	
"	37.0	E^1 I	..181 846	238 151 846	
"	38.9	F^1 II	918 048	919 048	
"	39.7	F^1 II	933 497	933 499	D
9—	42.7	E^1	obscure	0.136 899 089 179	N
10(231)	45.7	E^1 I	... 472	286 398 672	N
"	47.7	E^1	... 313	0.122 726 812 303	
"	49.3	F^1 III	3431	3437	
11(232)	50.6	E^1 I	..845 296	309 845 296	
11(232)	54.8	F^1 III	5181	5180	
12(233)	55.5	F^1 II	1 455 871	1 445 871	
"	59.9	F^1 II	... 165	1 737 166	D
"	"	F^1 III	8037	8035	
14(235)	66.5	E^1 II	62 603	62 601	D
15(236)	72.4	E^1 IV	53	63	
16(237)	72.1	F^1 IV	551	550	
"	74.5	F^1 I	955 ...	953 829 772	N
18(239)	79.2	F^1 III	... 975	144 976	D
19(240)	80.1	E^1 II	... 313	1 309 312	D
"	84.4	E^1	0,006 ...	0.006 613 376 378.88	
20(241)	84.7	F^1 IV	64 678	64 681	
21(242)	86.0	E^1	... 616 ...	0.003 739 613 163.78 *	N
22(243)	89.2	F^1 III	429 ...	431 285 857	D

ERRATA FOR PEARSON'S TABLE II (LEGENDRE'S TABLE IX)

PAGE	FUNCTION	φ	θ	FOR	READ		
24 (293)	F	32	3	05004	0.55858 01004		
"	F	42	2	09968	0.73311 00968	*	
25 (294)	E	51	4	88952	0.88962 98250	*	
26 (295)	F	64	3	64382	1.11750 63482		N
27 (296)	E	5	10	00237	0.08726 31277	*	N
"	E	41	10	98746	0.71391 97846		
"	E	44	6	15513	0.76721 15513		
28 (297)	F	5	7	08729	0.08726 81052	*	
"	F	20	9	34293	0.34923 53662	*	
29 (298)	E	90	8	56296	1.56316 22295	*	
30—	F	54	7	obscure	0.94421 96925		N
"	F	67	5	obscure	1.17091 31870		N
30 (299)	F	73	5	58448	1.27598 56448	*	N
31 (300)	E	7	11	12215	0.12216 20142		N
"	E	41	10		See above		
32 (301)	F	1	14	01743	0.01745 33444	*	N
"	F	6	14	11473	0.10473 09355		N
33 (302)	E	54	15	93450	0.93460 23938	*	N
33—	E	72	10	obscure	1.24934 56503		N
34 (303)	F	89	12	40839	1.57035 40439		
35 (304)	E	14	20	47569	0.24406 47569		
36 (305)	F	32	20	72078	0.56174 52078	*	
37 (306)	E	45	16	47234	0.77994 47334		N
39 (308)	E	14	20		See above		
"	E	20	24	80029	0.34791 80019		
"	E	40	24	08235	0.68953 08535		
41 (310)	E	50	20	obscure	0.86142 06177		N
"	E	62	23	obscure	1.05610 64197		N
"	E	78	22	31072	1.31972 85082	*	
42 (311)	F	88	21	58584	1.58784 47725	*	
"	F	90	23	63651	1.63651 74093	*	
44 (313)	F	35	30	62803	0.62003 01811	*	N
"	F	42	27	74754	0.74574 01816	*	
46 (315)	F	59	27	59233	1.06251 29233	*	
"	F	85	27	56840	1.56480 67133	*	
"	F	89	26	86937	1.63627 85937		
47 (316)	E	41	34	33784	0.69797 33984		
48 (317)	F	12	31	42285	0.20984 42245		N
49 (318)	E	48	33	56955	0.81181 56965		
50 (319)	F	84	34	05543	1.59518 05543		
51 (320)	E	45	36	72625	0.76003 72625		
53 (322)	E	45	35	76188	0.76128 30381	*	N
"	E	69	40	86991	1.10811 86891		
55 (324)	E	27	42	46376	0.46366 10915	*	
57 (326)	E	69	40		See above		
"	E	73	45	83451	1.13785 43451	*	
58 (327)	F	80	44	58906	1.59806 49280	*	
61 (330)	E	73	45		See above	*	
62 (331)	F	51	46	95158	0.95157 7953	*	
"	F	90	46	7545	1.86914 7546		

ERRATA FOR PEARSON'S TABLE II (LEGENDRE'S TABLE IX)—*continued*

PAGE	FUNCTION	φ	θ	FOR	READ		
70(339)	F	60	60	21253	1.21259 6614	*	
"	F	65	59	34195	1.34196 7808	*	
"	F	90	57	8167	2.08035 8067	*	
73(342)	E	90	64	9383	1.17317 9383		
74(343)	F	60	60		See above	*	
"	F	79	64	84693	1.84793 8718	*	
78(347)	F	89	66	5177	2.30100 5157		
(4pp.)	E	45	75	3043	0.71289 3045		
80(349)	F	37	75	69130	0.69190 9761	*	N
82(351)	F	61	71	29179	1.29719 6493	*	
"	F	85	74	7101	2.39615 6101		
"	F	84–88	75	Increase by 1×10^{-9}.			
83(352)	E	32	79	7384	0.53101 1384	*	
86(355)	F	50	79	2141	1.00317 2041		
"	F	56	76	3555	1.16468 3376		
"	F	55–89	76	See text			
"	F	90	78	8958	2.97856 8951		
88(357)	F	16	83	5780	0.28289 7580		
"	F	18	84	31936	0.31939 7699	*	
"	F	31	83	56907	0.56907 5915		N
90(359)	F	74	83	92585	1.92515 6351	*	
91(360)	E	6	86	10462	0.10452 9396		N
91—	E	21	85	obscure	0.35843 1155		N
92(361)	F	4	88	06936	0.06986 9880	*	N
"	F	31	90	55956	0.56956 2733	*	N
94(363)	F	49	87	98238	0.98328 1667	*	
"	F	86	86	204 1744	3.17030 9981	*	
"	F	87	87	44 5172	3.45667 6092	*	

f

$\cos^2\varphi$	$k'^2/k^2 = -.005$	0	.005	.01	.015	.02
−.005	.24961 01196	.24953 25478	.24945 47794	.24937 68196	.24929 86739	.24922 03473
0	.25007 80264	.25000 00000	.24992 17782	.24984 33666	.24976 47702	.24968 59944
.005	.25054 85410	.25047 00564	.25039 13779	.25031 25108	.25023 34605	.25015 42321
.01	.25102 16890	.25094 27428	.25086 36040	.25078 42781	.25070 47703	.25062 50859
.015	.25149 74969	.25141 80854	.25133 84828	.25125 86945	.25117 87258	.25109 85818
.02	.25197 59911	.25189 61108	.25181 60408	.25173 57867	.25165 53535	.25157 47464
.025	.25245 71987	.25237 68459	.25229 63050	.25221 55813	.25213 46801	.25205 36064
.03	.25294 11471	.25286 03183	.25277 93027	.25269 81058	.25261 67329	.25253 51890
.035	.25342 78642	.25334 65555	.25326 50616	.25318 33879	.25310 15396	.25301 95217
.04	.25391 73781	.25383 55858	.25375 36099	.25367 14556	.25358 91281	.25350 66328
.045	.25440 97175	.25432 74379	.25424 49761	.25416 23374	.25407 95272	.25399 65504
.05	.25490 49116	.25482 21407	.25473 91892	.25465 60624	.25457 27655	.25448 93036
.055	.25540 29899	.25531 97238	.25523 62787	.25515 26598	.25506 88724	.25498 49217
.06	.25590 39823	.25582 02171	.25573 62745	.25565 21597	.25556 78779	.25548 34343
.065	.25640 79192	.25632 36510	.25623 92068	.25615 45921	.25606 98120	.25598 48717
.07	.25691 48317	.25683 00563	.25674 51067	.25665 99880	.25657 47056	.25648 92645
.075	.25742 47510	.25733 94644	.25725 40052	.25716 83786	.25708 25398	.25699 66439
.08	.25793 77091	.25785 19071	.25776 59342	.25767 97955	.25759 34963	.25750 70416
.085	.25845 37383	.25836 74168	.25828 09259	.25819 42710	.25810 74572	.25802 04896
.09	.25897 28715	.25888 60262	.25879 90132	.25871 18379	.25862 45053	.25853 70206
.095	.25949 51421	.25940 77687	.25932 02294	.25923 25293	.25914 46737	.25905 66677
.10	.26002 05841	.25993 26782	.25984 46082	.25975 63791	.25966 79962	.25957 94645
.105	.26054 92319	.26046 07892	.26037 21840	.26028 34215	.26019 45069	.26010 54453
.11	.26108 11205	.26099 21365	.26090 29917	.26081 36914	.26072 42408	.26063 46448
.115	.26161 62855	.26152 67556	.26143 70668	.26134 72242	.26125 72331	.26116 70983
.12	.26215 47631	.26206 46828	.26197 44454	.26188 40560	.26179 35197	.26170 28417
.125	.26269 65901	.26260 59547	.26251 51640	.26242 42232	.26233 31373	.26224 19114
.13	.26324 18038	.26315 06086	.26305 92600	.26296 77631	.26287 61230	.26278 43446
.135	.26379 04421	.26369 86825	.26360 67711	.26351 47134	.26342 25143	.26333 01789
.14	.26434 25437	.26425 02147	.26415 77359	.26406 51126	.26397 23498	.26387 94525
.145	.26489 81478	.26480 52446	.26471 21935	.26461 89998	.26452 56684	.26443 22044
.15	.26545 72943	.26536 38119	.26527 01836	.26517 64145	.26508 25097	.26498 84742
.155	.26602 00238	.26592 59572	.26583 17466	.26573 73972	.26564 29141	.26554 83021
.16	.26658 63775	.26649 17216	.26639 69237	.26630 19890	.26620 69225	.26611 17291

θ

$\cos^2\varphi$	$k'^2/k^2 = -.005$	0	.005	.01	.015	.02
−.005	.50031 62190	.49937 65576	.49844 35293	.49751 70532	.49659 70498	.49568 34409
0	.50094 08340	.50000 00000	.49906 58068	.49813 81733	.49721 70201	.49630 22688
.005	.50156 85803	.50062 65674	.49969 12030	.49876 24061	.49784 00970	.49692 41974
.01	.50219 94873	.50125 62893	.50031 97476	.49938 97811	.49846 63099	.49754 92558
.015	.50283 35852	.50188 91957	.50095 14704	.50002 03280	.49909 56886	.49817 74738
.02	.50347 09042	.50252 53169	.50158 64016	.50065 40770	.49972 82631	.49880 88814
.025	.50411 14754	.50316 46837	.50222 45720	.50129 10588	.50036 40641	.49944 35092
.03	.50475 53298	.50380 73273	.50286 60128	.50193 13046	.50100 31226	.50008 13882
.035	.50540 24993	.50445 32794	.50351 07555	.50257 48458	.50164 54701	.50072 25498
.04	.50605 30161	.50510 25722	.50415 88322	.50322 17144	.50229 11386	.50136 70257
.045	.50670 69128	.50575 52381	.50481 02754	.50387 19430	.50294 01603	.50201 48485
.05	.50736 42227	.50641 13104	.50546 51182	.50452 55644	.50359 25683	.50266 60509
.055	.50802 49792	.50707 08225	.50612 33941	.50518 26121	.50424 83958	.50332 06662
.06	.50868 92167	.50773 38086	.50678 51371	.50584 31200	.50490 76768	.50397 87281
.065	.50935 69697	.50840 03032	.50745 03816	.50650 71226	.50557 04456	.50464 02710
.07	.51002 82733	.50907 03414	.50811 91627	.50717 46548	.50623 67370	.50530 53297
.075	.51070 31634	.50974 39589	.50879 15159	.50784 57521	.50690 65865	.50597 39395
.08	.51138 16760	.51042 11917	.50946 74774	.50852 04505	.50758 00300	.50664 61362
.085	.51206 38480	.51110 20766	.51014 70836	.50919 87864	.50825 71039	.50732 19563
.09	.51274 97168	.51178 66509	.51083 03719	.50988 07972	.50893 78454	.50800 14367
.095	.51343 93202	.51247 49524	.51151 73800	.51056 65203	.50962 22920	.50868 46150
.10	.51413 26968	.51316 70195	.51220 81462	.51125 59942	.51031 04819	.50937 15292
.105	.51482 98856	.51386 28912	.51290 27095	.51194 92576	.51100 24538	.51006 22181
.11	.51553 09264	.51456 26072	.51360 11094	.51264 63500	.51169 82473	.51075 67210
.115	.51623 58595	.51526 62077	.51430 33861	.51334 73115	.51239 79021	.51145 50777
.12	.51694 47259	.51597 37336	.51500 95804	.51405 21828	.51310 14591	.51215 73289
.125	.51765 75671	.51668 52264	.51571 97336	.51476 10053	.51380 89595	.51286 35156
.13	.51837 44255	.51740 07284	.51643 38880	.51547 38209	.51452 04451	.51357 36798
.135	.51909 53440	.51812 02823	.51715 20863	.51619 06725	.51523 59586	.51428 78640
.14	.51982 03663	.51884 39318	.51787 43720	.51691 16033	.51595 55433	.51500 61113
.145	.52054 95367	.51957 17211	.51860 07892	.51763 66574	.51667 92432	.51572 84657
.15	.52128 29003	.52030 36951	.51933 13829	.51836 58796	.51740 71030	.51645 49719
.155	.52202 05030	.52103 98997	.52006 61986	.51909 93155	.51813 91680	.51718 56750
.16	.52276 23912	.52178 03813	.52080 52827	.51983 70114	.51887 54847	.51792 06212

f

$\cos^2\varphi$	$h'^2/l^2 = .02$		.025		.03		.035		.04	
−.005	.24922	03473	.24914	18448	.24906	31716	.24898	43325	.24890	53322
0	.24968	59944	.24960	70442	.24952	79245	.24944	86402	.24936	91961
.005	.25015	42321	.25007	48306	.24999	52611	.24991	55284	.24983	56371
.01	.25062	50859	.25054	52298	.25046	52070	.25038	50224	.25030	46807
.015	.25109	85818	.25101	82676	.25093	77881	.25085	71482	.25077	63525
.02	.25157	47464	.25149	39706	.25141	30308	.25133	19321	.25125	06790
.025	.25205	36064	.25197	23654	.25189	09619	.25180	94008	.25172	76868
.03	.25253	51890	.25245	34792	.25237	16083	.25228	95813	.25220	74029
.035	.25301	95217	.25293	73395	.25285	49977	.25277	25012	.25268	98547
.04	.25350	66328	.25342	39744	.25334	11580	.25325	81883	.25317	50701
.045	.25399	65504	.25391	34122	.25383	01175	.25374	66710	.25366	30774
.05	.25448	93036	.25440	56818	.25432	19050	.25423	79779	.25415	39052
.055	.25498	49217	.25490	08124	.25481	65497	.25473	21382	.25464	75827
.06	.25548	34343	.25539	88338	.25531	40813	.25522	91816	.25514	41394
.065	.25598	48717	.25589	97760	.25581	45298	.25572	91381	.25564	36054
.07	.25648	92645	.25640	36697	.25631	79260	.25623	20382	.25614	60110
.075	.25699	66439	.25691	05460	.25682	43007	.25673	79130	.25665	13874
.08	.25750	70416	.25742	04364	.25733	36856	.25724	67938	.25715	97659
.085	.25802	04896	.25793	33731	.25784	61126	.25775	87128	.25767	11784
.09	.25853	70206	.25844	93886	.25836	16143	.25827	37023	.25818	56573
.095	.25905	66677	.25896	85160	.25888	02237	.25879	17953	.25870	32357
.10	.25957	94645	.25949	07889	.25940	19743	.25931	30254	.25922	39468
.105	.26010	54453	.26001	62415	.25992	69003	.25983	74266	.25974	78248
.11	.26063	46448	.26054	49084	.26045	50363	.26036	50334	.26027	49042
.115	.26116	70983	.26107	68249	.26098	64175	.26089	58810	.26080	52199
.12	.26170	28417	.26161	20267	.26152	10796	.26143	00051	.26133	88078
.125	.26224	19114	.26215	05505	.26205	90591	.26196	74420	.26187	57040
.13	.26278	43446	.26269	24330	.26260	03927	.26250	82287	.26241	59453
.135	.26333	01789	.26323	77119	.26314	51182	.26305	24024	.26295	95692
.14	.26387	94525	.26378	64255	.26369	32736	.26360	00015	.26350	66137
.145	.26443	22044	.26433	86126	.26424	48978	.26415	10646	.26405	71176
.15	.26498	84742	.26489	43128	.26480	00302	.26470	56311	.26461	11200
.155	.26554	83021	.26545	35662	.26535	87109	.26526	37111	.26516	86612
.16	.26611	17291	.26601	64136	.26592	09808	.26582	54353	.26572	97816

e

$\cos^2\varphi$	$k'^2/k^2 = .02$	.025	.03	.035	.04
−.005	.49568 34409	.49477 61497	.49387 51009	.49298 02200	.49209 14342
0	.49630 22688	.49539 38427	.49449 16661	.49359 56647	.49270 57656
.005	.49692 41974	.49601 46302	.49511 13200	.49421 41922	.49332 31737
.01	.49754 92558	.49663 85416	.49573 40916	.49483 58314	.49394 36877
.015	.49817 74738	.49726 56064	.49636 00106	.49546 06119	.49456 73369
.02	.49880 88814	.49789 58546	.49698 91069	.49608 85636	.49519 41513
.025	.49944 35092	.49852 93168	.49762 14110	.49671 97169	.49582 41611
.03	.50008 13882	.49916 60239	.49825 69536	.49735 41025	.49645 73971
.035	.50072 25498	.49980 60071	.49889 57661	.49799 17517	.49709 38904
.04	.50136 70257	.50044 92984	.49953 78802	.49863 26961	.49773 36726
.045	.50201 48485	.50109 59299	.50018 33281	.49927 69680	.49837 67758
.05	.50266 60509	.50174 59344	.50083 21425	.49992 45998	.49902 32326
.055	.50332 06662	.50239 93452	.50148 43565	.50057 56247	.49967 30760
.06	.50397 87281	.50305 61960	.50214 00038	.50123 00764	.50032 63394
.065	.50464 02710	.50371 65210	.50279 91187	.50188 79887	.50098 30570
.07	.50530 53297	.50438 03548	.50346 17356	.50254 93965	.50164 32632
.075	.50597 39395	.50504 77329	.50412 78898	.50321 43347	.50230 69931
.08	.50664 61362	.50571 86909	.50479 76171	.50388 28390	.50297 42823
.085	.50732 19563	.50639 32652	.50547 09536	.50455 49456	.50364 51668
.09	.50800 14367	.50707 14927	.50614 79362	.50523 06912	.50431 96833
.095	.50868 46150	.50775 34109	.50682 86023	.50591 01132	.50499 78691
.10	.50937 15292	.50843 90577	.50751 29897	.50659 32494	.50567 97620
.105	.51006 22181	.50912 84717	.50820 11372	.50728 01383	.50636 54003
.11	.51075 67210	.50982 16923	.50889 30837	.50797 08189	.50705 48230
.115	.51145 50777	.51051 87592	.50958 88691	.50866 53309	.50774 80697
.12	.51215 73289	.51121 97130	.51028 85337	.50936 37146	.50844 51807
.125	.51286 35156	.51192 45946	.51099 21186	.51006 60110	.50914 61967
.13	.51357 36798	.51263 34459	.51169 96653	.51077 22616	.50985 11594
.135	.51428 78640	.51334 63093	.51241 12164	.51148 25087	.51056 01108
.14	.51500 61113	.51406 32279	.51312 68148	.51219 67952	.51127 30938
.145	.51572 84657	.51478 42455	.51384 65041	.51291 51649	.51199 01520
.15	.51645 49719	.51550 94066	.51457 03290	.51363 76619	.51271 13296
.155	.51718 56750	.51623 87566	.51529 83344	.51436 43313	.51343 66716
.16	.51792 06212	.51697 23413	.51603 05663	.51509 52191	.51416 62237

f

$\cos^2\varphi$	$k'^2/k^2 = .04$	.045	.05	.055	.06
−.005	.24890 53322	.24882 61752	.24874 68665	.24866 74103	.24858 78110
0	.24936 01061	.24928 95069	.24920 98470	.24912 99510	.24904 99134
.005	.24983 56371	.24975 55921	.24967 53978	.24959 50588	.24951 45794
.01	.25030 46807	.25022 41865	.25014 35444	.25006 27590	.24998 18346
.015	.25077 63525	.25069 54058	.25061 43126	.25053 30774	.25045 17045
.02	.25125 06790	.25116 92763	.25108 77285	.25100 60400	.25092 42153
.025	.25172 76868	.25164 58246	.25156 38187	.25148 16735	.25139 93935
.03	.25220 74029	.25212 50776	.25204 26101	.25196 00048	.25187 72660
.035	.25268 98547	.25260 70628	.25252 41301	.25244 10611	.25235 73599
.04	.25317 50701	.25309 18080	.25300 84065	.25292 48701	.25284 12031
.045	.25366 30774	.25357 93414	.25349 54675	.25341 14601	.25332 73236
.05	.25415 39052	.25406 96916	.25398 53416	.25390 08595	.25381 62498
.055	.25464 75827	.25456 28877	.25447 80578	.25439 30974	.25430 80108
.06	.25514 41394	.25505 89593	.25497 36457	.25488 82032	.25480 26360
.065	.25564 36054	.25555 79362	.25547 21352	.25538 62068	.25530 01551
.07	.25614 60110	.25605 98490	.25597 35567	.25588 71384	.25580 05985
.075	.25665 13874	.25656 47286	.25647 79410	.25639 10290	.25630 39969
.08	.25715 97659	.25707 26063	.25698 53194	.25689 79098	.25681 03816
.085	.25767 11784	.25758 35139	.25749 57238	.25740 78125	.25731 97843
.09	.25818 56573	.25809 74839	.25800 91865	.25792 07695	.25783 22372
.095	.25870 32357	.25861 45492	.25852 57404	.25843 68136	.25834 77730
.10	.25922 39468	.25913 47431	.25904 54187	.25895 59779	.25886 64251
.105	.25974 78248	.25965 80996	.25956 82554	.25947 82964	.25938 82271
.11	.26027 49042	.26018 46532	.26009 42848	.26000 38035	.25991 32134
.115	.26080 52199	.26071 44383	.26062 35420	.26053 25340	.26044 14188
.12	.26133 88078	.26124 74921	.26115 60625	.26106 45234	.26097 28788
.125	.26187 57040	.26178 38493	.26169 18825	.26159 98078	.26150 76295
.13	.26241 59453	.26232 35471	.26223 10385	.26213 84239	.26204 57073
.135	.26295 95692	.26286 66230	.26277 35680	.26268 04088	.26258 71495
.14	.26350 66137	.26341 31148	.26331 95089	.26322 58006	.26313 19939
.145	.26405 71176	.26396 30612	.26386 88998	.26377 46376	.26368 02790
.15	.26461 11200	.26451 65015	.26442 17797	.26432 69591	.26423 20438
.155	.26516 86612	.26507 34756	.26497 81887	.26488 28048	.26478 73280
.16	.26572 97816	.26563 40241	.26553 81672	.26544 22152	.26534 61722

e

$\cos^2\varphi$	$k'^2/k^2 = .04$	.045	.05	.055	.06
−.005	.49209 14342	.49120 86718	.49033 18620	.48946 09356	.48859 58242
0	.49270 57656	.49182 18967	.49094 39875	.49007 19686	.48920 57715
.005	.49332 31737	.49243 81926	.49155 91782	.49068 60609	.48981 87723
.01	.49394 36877	.49305 75885	.49217 74629	.49130 32415	.49043 48556
.015	.49456 73369	.49368 01136	.49279 88710	.49192 35395	.49105 40505
.02	.49519 41513	.49430 57978	.49342 34322	.49254 69847	.49167 63867
.025	.49582 41611	.49493 46714	.49405 11767	.49317 36072	.49230 18941
.03	.49645 73971	.49556 67650	.49468 21351	.49380 34375	.49293 06033
.035	.49709 38904	.49620 21097	.49531 63384	.49443 65065	.49356 25451
.04	.49773 36726	.49684 07370	.49595 38181	.49507 28458	.49419 77510
.045	.49837 67758	.49748 26790	.49659 46062	.49571 24871	.49483 62528
.05	.49902 32326	.49812 79682	.49723 87350	.49635 54630	.49547 80828
.055	.49967 30760	.49877 66374	.49788 62376	.49700 18061	.49612 32737
.06	.50032 63394	.49942 87202	.49853 71472	.49765 15498	.49677 18588
.065	.50098 30570	.50008 42506	.49919 14977	.49830 47279	.49742 38718
.07	.50164 32632	.50074 32628	.49984 93236	.49896 13747	.49807 93470
.075	.50230 69831	.50140 57921	.50051 06597	.49962 15252	.49873 83191
.08	.50297 42823	.50207 18737	.50117 55414	.50028 52145	.49940 08235
.085	.50364 51668	.50274 15439	.50184 40048	.50095 24787	.50006 68959
.09	.50431 96833	.50341 48391	.50251 60863	.50162 33542	.50073 65728
.095	.50499 78691	.50409 17965	.50319 18231	.50229 78779	.50140 98910
.10	.50567 97620	.50477 24538	.50387 12527	.50297 60874	.50208 68881
.105	.50636 54003	.50545 68494	.50455 44134	.50365 80210	.50276 76022
.11	.50705 48230	.50614 50222	.50524 13441	.50434 37173	.50345 20719
.115	.50774 80697	.50683 70116	.50593 20841	.50503 32158	.50414 03364
.12	.50844 51807	.50753 28578	.50662 66736	.50572 65563	.50483 24358
.125	.50914 61967	.50823 26017	.50732 51531	.50642 37795	.50552 84105
.13	.50985 11594	.50893 62845	.50802 75642	.50712 49267	.50622 83018
.135	.51056 01108	.50964 39484	.50873 39487	.50783 00398	.50693 21513
.14	.51127 30938	.51035 56362	.50944 43493	.50853 91614	.50764 00018
.145	.51199 01520	.51107 13912	.51015 88004	.50925 23347	.50835 18962
.15	.51271 13206	.51179 12578	.51087 73732	.50996 96037	.50906 78787
.155	.51343 66716	.51251 52807	.51160 00853	.51069 10132	.50978 79937
.16	.51416 62237	.51324 35056	.51232 69913	.51141 66086	.51051 22866

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$\cos^2\varphi$	$k'^2/k^2 = .06$	.065	.07	.075	.08
−.005	.24858 78110	.24850 80729	.24842 82005	.24834 81975	.24826 80682
0	.24904 99134	.24896 97382	.24888 94299	.24880 89925	.24872 84300
.005	.24951 45794	.24943 39639	.24935 32165	.24927 23414	.24919 13425
.01	.24998 18346	.24990 07753	.24981 95856	.24973 82694	.24965 68308
.015	.25045 17045	.25037 01982	.25028 85627	.25020 68022	.25012 49206
.02	.25092 42153	.25084 22586	.25076 01740	.25067 79657	.25059 56377
.025	.25139 93935	.25131 69828	.25123 44458	.25115 17863	.25106 90086
.03	.25187 72660	.25179 43979	.25171 14048	.25162 82908	.25154 50599
.035	.25235 78599	.25227 45310	.25219 10784	.25210 75063	.25202 38187
.04	.25284 12031	.25275 74097	.25267 34941	.25258 94604	.25250 53125
.045	.25332 73236	.25324 30621	.25315 86798	.25307 41809	.25298 95693
.05	.25381 62498	.25373 15166	.25364 66641	.25356 16964	.25347 66175
.055	.25430 80108	.25422 28022	.25413 74758	.25405 20356	.25396 64857
.06	.25480 26360	.25471 69482	.25463 11442	.25454 52278	.25445 92032
.065	.25530 01551	.25521 39844	.25512 76990	.25504 13027	.25495 47996
.07	.25580 05985	.25571 39411	.25562 71704	.25554 02905	.25545 33052
.075	.25630 39969	.25621 68489	.25612 95892	.25604 22216	.25595 47503
.08	.25681 03816	.25672 27391	.25663 49863	.25654 71274	.25645 91662
.085	.25731 97843	.25723 16433	.25714 33936	.25705 50393	.25696 65843
.09	.25783 22372	.25774 35936	.25765 48431	.25756 59894	.25747 70366
.095	.25834 77730	.25825 86229	.25816 93673	.25808 00103	.25799 05557
.10	.25886 64251	.25877 67642	.25868 69996	.25859 71351	.25850 71746
.105	.25938 82271	.25929 80514	.25920 77735	.25911 73974	.25902 69270
.11	.25991 32134	.25982 25186	.25973 17233	.25964 08314	.25954 98468
.115	.26044 14188	.26035 02007	.26025 88837	.26016 74718	.26007 59688
.12	.26097 28788	.26088 11330	.26078 92900	.26069 73538	.26060 53282
.125	.26150 76295	.26141 53516	.26132 29782	.26123 05133	.26113 79608
.13	.26204 57073	.26195 28929	.26185 99847	.26176 69868	.26167 39029
.135	.26258 71495	.26249 37941	.26240 03467	.26230 68112	.26221 31915
.14	.26313 19939	.26303 80929	.26294 41017	.26285 00242	.26275 58842
.145	.26368 02790	.26358 58278	.26349 12882	.26339 66641	.26330 19592
.15	.26423 20438	.26413 70377	.26404 19451	.26394 67697	.26385 15154
.155	.26478 73280	.26469 17624	.26459 61120	.26450 03807	.26440 45722
.16	.26534 61722	.26525 00422	.26515 38292	.26505 75372	.26496 11698

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$\cos^2\varphi$	$k'^2/k^2 = .06$		.065		.07		.075		.08	
−.005	.48859	58242	.48773	64607	.48688	27791	.48603	47143	.48519	22025
0	.48920	57715	.48834	53290	.48749	05750	.48664	14445	.48579	78734
.005	.48981	87723	.48895	72452	.48810	14132	.48725	12112	.48640	65753
.01	.49043	48556	.48957	22380	.48871	53222	.48786	40432	.48701	83367
.015	.49105	40505	.49019	03365	.48933	23312	.48847	99693	.48763	31866
.02	.49167	63867	.49081	15705	.48995	24608	.48909	90192	.48825	11545
.025	.49230	18941	.49143	59698	.49057	57678	.48972	12227	.48887	22701
.03	.49293	06033	.49206	35648	.49120	22555	.49034	66099	.48949	65636
.035	.49356	25451	.49269	43865	.49183	19639	.49097	52119	.49012	40658
.04	.49419	77510	.49332	84661	.49246	49241	.49160	70596	.49075	48078
.045	.49483	62528	.49396	58353	.49310	11679	.49224	21847	.49138	88211
.05	.49547	80828	.49460	65265	.49374	07273	.49288	06194	.49202	61379
.055	.49612	32737	.49525	05724	.49438	36351	.49352	23961	.49266	67905
.06	.49677	18588	.49589	80060	.49502	99244	.49416	75481	.49331	08121
.065	.49742	38718	.49654	88611	.49567	96288	.49481	61087	.49395	82360
.07	.49807	93470	.49720	31719	.49633	27823	.49546	81121	.49460	90963
.075	.49873	83191	.49786	09730	.49698	94197	.49612	35929	.49526	34275
.08	.49940	08235	.49852	22998	.49764	95760	.49678	25860	.49592	12646
.085	.50006	68959	.49918	71878	.49831	32871	.49744	51272	.49658	26431
.09	.50073	65728	.49985	56736	.49898	05890	.49811	12526	.49724	75991
.095	.50140	98910	.50052	77938	.49965	15186	.49878	09989	.49791	61693
.10	.50208	68881	.50120	35860	.50032	61132	.49945	44033	.49858	83908
.105	.50276	76022	.50188	30880	.50100	44108	.50013	15038	.49926	43015
.11	.50345	20719	.50256	63387	.50168	64499	.50081	23388	.49994	39397
.115	.50414	03364	.50325	33770	.50237	22696	.50149	69473	.50062	73445
.12	.50483	24358	.50394	42429	.50306	19097	.50218	53690	.50131	45553
.125	.50552	84105	.50463	89769	.50375	54104	.50287	76443	.50200	56124
.13	.50622	83018	.50533	76199	.50445	28130	.50357	38139	.50270	05567
.135	.50693	21513	.50604	02138	.50515	41589	.50427	39196	.50339	94296
.14	.50764	00018	.50674	68010	.50585	94906	.50497	80035	.50410	22734
.145	.50835	18962	.50745	74245	.50656	88511	.50568	61087	.50480	91309
.15	.50906	78787	.50817	21283	.50728	22841	.50639	82787	.50552	00457
.155	.50978	79937	.50889	09568	.50799	98341	.50711	45579	.50623	50620
.16	.51051	22866	.50961	39554	.50872	15462	.50783	49915	.50695	42248

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$\cos^2\varphi$	$k'^2/k^2 = .08$	.085	.09	.095	.10
−.005	.24826 80682	.24818 78166	.24810 74464	.24802 69617	.24794 63660
0	.24872 84300	.24864 77465	.24856 69458	.24848 60317	.24840 50080
.005	.24919 13425	.24911 02238	.24902 89893	.24894 76428	.24886 61879
.01	.24965 68308	.24957 52738	.24949 36022	.24941 18199	.24932 99306
.015	.25012 49206	.25004 29219	.24996 08100	.24987 85887	.24979 62616
.02	.25059 56377	.25051 31940	.25043 06384	.25034 79748	.25026 52067
.025	.25106 90086	.25098 61165	.25090 31138	.25082 00045	.25073 67921
.03	.25154 50599	.25146 17159	.25137 82628	.25129 47044	.25121 10443
.035	.25202 38187	.25194 00194	.25185 61125	.25177 21015	.25168 79902
.04	.25250 53125	.25242 10545	.25233 66901	.25225 22232	.25216 76573
.045	.25298 95693	.25290 48490	.25282 00237	.25273 50972	.25265 00733
.05	.25347 66175	.25339 14312	.25330 61414	.25322 07519	.25313 52663
.055	.25396 64857	.25388 08298	.25379 50720	.25370 92158	.25362 32649
.06	.25445 92032	.25437 30741	.25428 68445	.25420 05180	.25411 40983
.065	.25495 47996	.25486 81936	.25478 14885	.25469 46880	.25460 77958
.07	.25545 33052	.25536 62185	.25527 90341	.25519 17559	.25510 43874
.075	.25595 47503	.25586 71791	.25577 95118	.25569 17521	.25560 39036
.08	.25645 91662	.25637 11066	.25628 29524	.25619 47074	.25610 63751
.085	.25696 65843	.25687 80324	.25678 93875	.25670 06533	.25661 18333
.09	.25747 70366	.25738 79885	.25729 88490	.25720 96216	.25712 03101
.095	.25799 05557	.25790 10074	.25781 13692	.25772 16448	.25763 18377
.10	.25850 71746	.25841 71221	.25832 69812	.25823 67557	.25814 64491
.105	.25902 69270	.25893 63661	.25884 57184	.25875 49878	.25866 41776
.11	.25954 98468	.25945 87734	.25936 76149	.25927 63749	.25918 50571
.115	.26007 59688	.25998 43787	.25989 27051	.25980 09517	.25970 91221
.12	.26060 53282	.26051 32171	.26042 10242	.26032 87531	.26023 64075
.125	.26113 79608	.26104 53244	.26095 26079	.26085 98149	.26076 69490
.13	.26167 39029	.26158 07368	.26148 74924	.26139 41731	.26130 07826
.135	.26221 31915	.26211 94914	.26202 57146	.26193 18647	.26183 79452
.14	.26275 58642	.26266 16256	.26256 73119	.26247 29269	.26237 84741
.145	.26330 19592	.26320 71775	.26311 23225	.26301 73979	.26292 24072
.15	.26385 15154	.26375 61860	.26366 07851	.26356 53163	.26346 97832
.155	.26440 45722	.26430 86905	.26421 27390	.26411 67215	.26402 06414
.16	.26496 11698	.26486 47310	.26476 82243	.26467 16533	.26457 50216

e

$\cos^2\varphi$	$k'^2/k^2 = .08$	.085	.09	.095	.10
−.005	.48519 22025	.48435 51807	.48352 35870	.48269 73606	.48187 64414
0	.48579 78734	.48495 97988	.48412 71587	.48329 98921	.48247 79390
.005	.48640 65753	.48556 74423	.48473 37502	.48390 54379	.48308 24454
.01	.48701 83367	.48617 81397	.48534 33900	.48451 40265	.48368 99891
.015	.48763 31856	.48679 19199	.48595 61070	.48512 56867	.48430 05988
.02	.48825 11545	.48740 88123	.48657 19305	.48574 04477	.48491 43036
.025	.48887 22701	.48802 88466	.48719 08901	.48635 83391	.48553 11332
.03	.48949 65636	.48865 20531	.48781 30160	.48697 93910	.48615 11175
.035	.49012 40658	.48927 84623	.48843 83388	.48760 36339	.48677 42871
.04	.49075 48078	.48990 81053	.48906 68894	.48823 10988	.48740 06727
.045	.49138 88211	.49054 10136	.48969 86994	.48886 18170	.48803 03057
.05	.49202 61379	.49117 72192	.49033 38006	.48949 58204	.48866 32180
.055	.49266 67905	.49181 67545	.49097 22254	.49013 31414	.48929 94418
.06	.49331 08121	.49245 96526	.49161 40068	.49077 38128	.48993 90097
.065	.49395 82360	.49310 59467	.49225 91779	.49141 78677	.49058 19552
.07	.49460 90963	.49375 56709	.49290 77728	.49206 53401	.49122 83118
.075	.49526 34275	.49440 88595	.49355 98257	.49271 62642	.49187 81138
.08	.49592 12646	.49506 55475	.49421 53716	.49337 06748	.49253 13960
.085	.49658 26431	.49572 57703	.49487 44458	.49402 86073	.49318 81937
.09	.49724 75991	.49638 95641	.49553 70844	.49469 00976	.49384 85426
.095	.49791 61693	.49705 69653	.49620 33237	.49535 51821	.49451 24791
.10	.49858 83908	.49772 80111	.49687 32009	.49602 38977	.49518 00401
.105	.49926 43015	.49840 27393	.49754 67537	.49669 62821	.49585 12632
.11	.49994 39397	.49908 11880	.49822 40201	.49737 23734	.49652 61863
.115	.50062 73445	.49976 33963	.49890 50391	.49805 22103	.49720 48483
.12	.50131 45553	.50044 94036	.49958 98501	.49873 58323	.49788 72883
.125	.50200 56124	.50113 92500	.50027 84932	.49942 32792	.49857 35463
.13	.50270 05567	.50183 29764	.50097 10090	.50011 45918	.49926 36628
.135	.50339 94296	.50253 06241	.50166 74389	.50080 98112	.49995 76790
.14	.50410 22734	.50323 22353	.50236 78250	.50150 89795	.50065 56367
.145	.50480 91309	.50393 78527	.50307 22098	.50221 21391	.50135 75786
.15	.50552 00457	.50464 75198	.50378 06368	.50291 93336	.50206 35478
.155	.50623 50620	.50536 12808	.50449 31501	.50363 06067	.50277 35882
.16	.50695 42248	.50607 91806	.50520 97946	.50434 60033	.50348 77444

f

$\cos^2\varphi$	$k'^2/k^2 = .10$	.105	.11	.115	.12
−.005	.24794 63660	.24786 56630	.24778 48564	.24770 39497	.24762 29462
0	.24840 50080	.24832 38783	.24824 26462	.24816 13153	.24807 98889
.005	.24886 61879	.24878 46282	.24870 29676	.24862 12092	.24853 93568
.01	.24932 99306	.24924 79378	.24916 58453	.24908 36565	.24900 13748
.015	.24979 62616	.24971 38325	.24963 13049	.24954 86823	.24946 59681
.02	.25026 52067	.25018 23379	.25009 93720	.25001 63123	.24993 31625
.025	.25073 67921	.25065 34803	.25057 00727	.25048 65727	.25040 29838
.03	.25121 10443	.25112 72861	.25104 34335	.25095 94898	.25087 54586
.035	.25168 79902	.25160 37823	.25151 94813	.25143 50906	.25135 06136
.04	.25216 76573	.25208 29962	.25199 82433	.25191 34021	.25182 84761
.045	.25265 00733	.25256 49554	.25247 97472	.25239 44521	.25230 90735
.05	.25313 52663	.25304 96882	.25296 40211	.25287 82686	.25279 24340
.055	.25362 32649	.25353 72230	.25345 10936	.25336 48801	.25327 85859
.06	.25411 40983	.25402 75889	.25394 09935	.25385 43155	.25376 75582
.065	.25460 77958	.25452 08154	.25443 37504	.25434 66041	.25425 93800
.07	.25510 43874	.25501 69323	.25492 93939	.25484 17758	.25475 40813
.075	.25560 39036	.25551 59699	.25542 79545	.25533 98608	.25525 16922
.08	.25610 63751	.25601 79591	.25592 94629	.25584 08899	.25575 22434
.085	.25661 18333	.25652 29311	.25643 39503	.25634 48942	.25625 57661
.09	.25712 03101	.25703 09179	.25694 14485	.25685 19054	.25676 22918
.095	.25763 18377	.25754 19515	.25745 19898	.25736 19558	.25727 18529
.10	.25814 64491	.25805 60650	.25796 56068	.25787 50780	.25778 44818
.105	.25866 41776	.25857 32915	.25848 23329	.25839 13052	.25830 02118
.11	.25918 50571	.25909 36649	.25900 22019	.25891 06713	.25881 90765
.115	.25970 91221	.25961 72197	.25952 52481	.25943 32105	.25934 11103
.12	.26023 64075	.26014 39907	.26005 15063	.25995 89576	.25986 63479
.125	.26076 69490	.26067 40136	.26058 10122	.26048 79482	.26039 48247
.13	.26130 07826	.26120 73243	.26111 38017	.26102 02180	.26092 65767
.135	.26183 79452	.26174 39596	.26164 99114	.26155 58039	.26146 16402
.14	.26237 84741	.26228 39568	.26218 93787	.26209 47428	.26200 00526
.145	.26292 24072	.26282 73539	.26273 22413	.26263 70727	.26254 18515
.15	.26346 97832	.26337 41892	.26327 85377	.26318 28320	.26308 70753
.155	.26402 06414	.26392 45021	.26382 83071	.26373 20597	.26363 57630
.16	.26457 50216	.26447 83325	.26438 15894	.26428 47956	.26418 79543

e

$\cos^2\varphi$	$k'^2/k^2 = .10$	.105	.11	.115	.12
−.005	.48187 64414	.48106 07704	.48025 02896	.47944 49417	.47864 46705
0	.48247 79390	.48166 12403	.48084 97378	.48004 33744	.47924 20936
.005	.48308 24454	.48226 47135	.48145 21841	.48064 47997	.47984 25039
.01	.48368 99891	.48287 12186	.48205 76566	.48124 92458	.48044 59297
.015	.48430 05988	.48348 07840	.48266 61840	.48185 67413	.48105 23994
.02	.48491 42036	.48409 34390	.48327 77954	.48246 73153	.48166 19421
.025	.48553 11332	.48470 92131	.48389 25203	.48308 09971	.48227 45871
.03	.48615 11175	.48532 81362	.48451 03885	.48369 78167	.48289 03641
.035	.48677 42871	.48595 02 38	.48513 14304	.48431 78042	.48350 93035
.04	.48740 06727	.48657 55516	.48575 56767	.48494 09904	.48413 14358
.045	.48803 03057	.48720 41059	.48638 31587	.48556 74065	.48475 67921
.05	.48866 32180	.48783 59335	.48701 39081	.48619 70840	.48538 54041
.055	.48929 94418	.48847 10666	.48764 79570	.48683 00551	.48601 73037
.06	.48993 90097	.48910 95378	.48828 53380	.48746 63523	.48665 25235
.065	.49058 19552	.48975 13803	.48892 60842	.48810 60086	.48729 10964
.07	.49122 83118	.49039 66279	.48957 02292	.48874 90576	.48793 30558
.075	.49187 81138	.49104 53145	.49021 78071	.48939 55333	.48857 84359
.08	.49253 13960	.49169 74750	.49086 88526	.49004 54704	.48922 72710
.085	.49318 81937	.49235 31446	.49152 34007	.49069 89037	.48987 95961
.09	.49384 85426	.49301 23589	.49218 14872	.49135 58690	.49053 54468
.095	.49451 24791	.49367 51543	.49284 31482	.49201 64024	.49119 48592
.10	.49518 00401	.49434 15676	.49350 84206	.49268 05406	.49185 78700
.105	.49585 12632	.49501 16362	.49417 73417	.49334 83210	.49252 45163
.11	.49652 61863	.49568 53982	.49484 99495	.49401 97813	.49319 48359
.115	.49720 48483	.49636 28922	.49552 62824	.49469 49600	.49386 88672
.12	.49788 72883	.49704 41573	.49620 63796	.49537 38963	.49454 66493
.125	.49857 35463	.49772 92335	.49689 02809	.49605 66297	.49522 82217
.13	.49926 36628	.49841 81611	.49757 80267	.49674 32005	.49591 36246
.135	.49995 76790	.49911 09812	.49826 96579	.49743 36498	.49660 28989
.14	.50065 56367	.49980 77357	.49896 52163	.49812 80192	.49729 60862
.145	.50135 75786	.50050 84670	.49966 47442	.49882 63508	.49799 32286
.15	.50206 35478	.50121 32182	.50036 82847	.49952 86878	.49869 43691
.155	.50277 35882	.50192 20332	.50107 58816	.50023 50738	.49939 95513
.16	.50348 77444	.50263 49566	.50178 75793	.50094 55531	.50010 88195

f

$\cos^2\varphi$	$k'^2/k^2 = .12$	.125	.13	.135	.14
−.005	.24762 29462	.24754 18495	.24746 06628	.24737 93895	.24729 80325
0	.24807 98889	.24799 83705	.24791 67634	.24783 50708	.24775 32960
.005	.24853 93568	.24845 74136	.24837 53829	.24829 32680	.24821 10722
.01	.24900 13748	.24891 90036	.24883 65462	.24875 40059	.24867 13858
.015	.24946 59681	.24938 31657	.24930 02785	.24921 73096	.24913 42622
.02	.24993 31625	.24984 99257	.24976 66053	.24968 32046	.24959 97266
.025	.25040 29838	.25031 93093	.25023 55526	.25015 17168	.25006 78051
.03	.25087 54586	.25079 13431	.25070 71467	.25062 28725	.25053 85238
.035	.25135 06136	.25126 60538	.25118 14144	.25109 66985	.25101 19094
.04	.25182 84761	.25174 34685	.25165 83826	.25157 32217	.25148 79889
.045	.25230 90735	.25222 36147	.25213 80791	.25205 24697	.25196 67898
.05	.25279 24340	.25270 65205	.25262 05316	.25253 44704	.25244 83400
.055	.25327 85859	.25319 22143	.25310 57686	.25301 92520	.25293 26676
.06	.25376 75582	.25368 07249	.25359 38189	.25350 68433	.25341 98014
.065	.25425 93800	.25417 20814	.25408 47115	.25399 72735	.25390 97705
.07	.25475 40813	.25466 63138	.25457 84764	.25449 05722	.25440 26046
.075	.25525 16922	.25516 34520	.25507 51434	.25498 67696	.25489 83336
.08	.25575 22434	.25566 35268	.25557 47433	.25548 58960	.25539 69881
.085	.25625 57661	.25616 65693	.25607 73072	.25598 79827	.25589 85990
.09	.25676 22918	.25667 26111	.25658 28665	.25649 30610	.25640 31978
.095	.25727 18529	.25718 16843	.25709 14533	.25700 11630	.25691 08165
.10	.25778 44818	.25769 38215	.25760 31003	.25751 23213	.25742 14876
.105	.25830 02118	.25820 90557	.25811 78403	.25802 65687	.25793 52439
.11	.25881 90765	.25872 74208	.25863 57072	.25854 39389	.25845 21191
.115	.25934 11103	.25924 89507	.25915 67350	.25906 44660	.25897 21471
.12	.25986 63479	.25977 36804	.25968 09583	.25958 81847	.25949 53625
.125	.26039 48247	.26030 16451	.26020 84126	.26011 51300	.26002 18006
.13	.26092 65767	.26083 28808	.26073 91335	.26064 53379	.26055 14970
.135	.26146 16402	.26136 74237	.26127 31575	.26117 88446	.26108 44881
.14	.26200 00526	.26190 53112	.26181 05217	.26171 56872	.26162 08107
.145	.26254 18515	.26244 65808	.26235 12636	.26225 59032	.26216 05024
.15	.26308 70753	.26299 12708	.26289 54216	.26279 95308	.26270 36013
.155	.26363 57630	.26353 94202	.26344 30345	.26334 66089	.26325 01463
.16	.26418 79543	.26409 10687	.26399 41419	.26389 71769	.26380 01767

e

$\cos^2\varphi$	$k'^2/k^2 = .12$	.125	.13	.135	.14
−.005	.47864 46705	.47784 94207	.47705 91375	.47627 37673	.47549 32572
0	.47924 20936	.47844 58400	.47765 45590	.47686 81968	.47608 67004
.005	.47984 25039	.47904 52414	.47825 29572	.47746 55977	.47668 31098
.01	.48044 59297	.47964 76527	.47885 43601	.47806 59980	.47728 25133
.015	.48105 23994	.48025 31027	.47945 87962	.47866 94261	.47788 49392
.02	.48166 19421	.48086 16201	.48006 62943	.47927 59108	.47849 04164
.025	.48227 45871	.48147 32342	.48067 68837	.47988 54814	.47909 89740
.03	.48289 03641	.48208 79749	.48129 05940	.48049 81673	.47971 06415
.035	.48350 93035	.48270 58722	.48190 74554	.48111 39988	.48032 54490
.04	.48413 14358	.48332 69568	.48252 74983	.48173 30061	.48094 34268
.045	.48475 67921	.48395 12597	.48315 07539	.48235 52204	.48156 46058
.05	.48538 54041	.48457 88124	.48377 72534	.48298 06730	.48218 90174
.055	.48601 73037	.48520 96468	.48440 70289	.48360 93957	.48281 66934
.06	.48665 25235	.48584 37955	.48504 01127	.48424 14208	.48344 76659
.065	.48729 10964	.48648 12912	.48567 65377	.48487 67811	.48408 19678
.07	.48793 30558	.48712 21675	.48631 63371	.48551 55100	.48471 96323
.075	.48857 84359	.48776 64583	.48695 95449	.48615 76412	.48536 06930
.08	.48922 72710	.48841 41979	.48760 61955	.48680 32089	.48600 51843
.085	.48987 95961	.48906 54213	.48825 63236	.48745 22481	.48665 31408
.09	.49053 54468	.48972 01640	.48890 99647	.48810 47940	.48730 45978
.095	.49119 48592	.49037 84619	.48956 71547	.48876 08825	.48795 95912
.10	.49185 78700	.49104 03518	.49022 79302	.48942 05502	.48861 81574
.105	.49252 45163	.49170 58707	.49089 23283	.49008 38339	.48928 03332
.11	.49319 48359	.49237 50563	.49156 03865	.49075 07713	.48994 61563
.115	.49386 88672	.49304 79470	.49223 21432	.49142 14005	.49061 56646
.12	.49454 66493	.49372 45816	.49290 76371	.49209 57603	.49128 88969
.125	.49522 82217	.49440 49998	.49358 69077	.49277 38901	.49196 58924
.13	.49591 36246	.49508 92416	.49426 99952	.49345 58300	.49264 66912
.135	.49660 28989	.49577 73478	.49495 69402	.49414 16204	.49333 13338
.14	.49729 60862	.49646 93600	.49564 77841	.49483 13028	.49401 98614
.145	.49799 32286	.49716 53202	.49634 25689	.49552 49191	.49471 23160
.15	.49869 43691	.49786 52713	.49704 13374	.49622 25120	.49540 87400
.155	.49939 95513	.49856 92567	.49774 41331	.49692 41248	.49610 91768
.16	.50010 88195	.49927 73207	.49845 10001	.49762 98016	.49681 36704

f

$\cos^2\varphi$	$k'^2/k^2 = .14$	.145	.15	.155	.16
−.005	.24729 80325	.24721 65953	.24713 50807	.24705 34916	.24697 18314
0	.24775 32960	.24767 14420	.24758 95118	.24750 75087	.24742 54353
.005	.24821 10722	.24812 87984	.24804 64497	.24796 40292	.24788 15398
.01	.24867 13858	.24858 86891	.24850 59188	.24842 30779	.24834 01693
.015	.24913 42622	.24905 11394	.24896 79443	.24888 46798	.24880 13489
.02	.24959 97266	.24951 61746	.24943 25515	.24934 88604	.24926 51040
.025	.25006 78051	.24998 38206	.24989 97663	.24981 56453	.24973 14603
.03	.25053 85238	.25045 41036	.25036 96148	.25028 50607	.25020 04439
.035	.25101 19094	.25092 70501	.25084 21237	.25075 71331	.25067 20812
.04	.25148 79889	.25140 26873	.25131 73198	.25123 18895	.25114 63992
.045	.25196 67898	.25188 10424	.25179 52305	.25170 93572	.25162 34251
.05	.25244 83400	.25236 21434	.25227 58837	.25218 95638	.25210 31867
.055	.25293 26676	.25284 60184	.25275 93074	.25267 25377	.25258 57120
.06	.25341 98014	.25333 26961	.25324 55304	.25315 83073	.25307 10296
.065	.25390 97705	.25382 22056	.25373 45816	.25364 69016	.25355 91684
.07	.25440 26046	.25431 45764	.25422 64905	.25413 83501	.25405 01579
.075	.25489 83336	.25480 98385	.25472 12872	.25463 26828	.25454 40280
.08	.25539 69881	.25530 80224	.25521 90021	.25512 99300	.25504 08089
.085	.25589 85990	.25580 91591	.25571 96659	.25563 01225	.25554 05315
.09	.25640 31978	.25631 32799	.25622 33102	.25613 32916	.25604 32270
.095	.25691 08165	.25682 04168	.25672 99667	.25663 94633	.25654 89273
.10	.25742 14876	.25733 06021	.25723 96679	.25714 86878	.25705 76646
.105	.25793 52439	.25784 38689	.25775 24466	.25766 09799	.25756 94717
.11	.25845 21191	.25836 02505	.25826 83362	.25817 63791	.25808 43818
.115	.25897 21471	.25887 97810	.25878 73707	.25869 49191	.25860 24290
.12	.25949 53625	.25940 24949	.25930 95846	.25921 66345	.25912 36475
.125	.26002 18006	.25992 84273	.25983 50128	.25974 15603	.25964 80722
.13	.26055 14970	.26045 76138	.26036 36912	.26026 97319	.26017 57388
.135	.26108 44881	.26099 00909	.26089 56557	.26080 11857	.26070 66834
.14	.26162 08107	.26152 58952	.26143 09434	.26133 59582	.26124 09425
.145	.26216 05024	.26206 50642	.26196 95915	.26187 40870	.26177 85535
.15	.26270 36013	.26260 76362	.26251 16380	.26241 56099	.26231 95544
.155	.26325 01463	.26315 36497	.26305 71218	.26296 05656	.26286 39837
.16	.26380 01767	.26370 31442	.26360 60821	.26350 89934	.26341 18806

θ

$\cos^2\varphi$	$k'^2/k^2 = .14$	.145	.15	.155	.16
−.005	.47549 32572	.47471 75551	.47394 66096	.47318 03703	.47241 87873
0	.47608 67004	.47531 00176	.47453 80972	.47377 08884	.47300 83414
.005	.47668 31098	.47590 54412	.47513 25405	.47436 43571	.47360 08412
.01	.47728 25133	.47650 38536	.47572 99676	.47496 08044	.47419 63142
.015	.47788 49392	.47710 52832	.47633 04065	.47556 02583	.47479 47887
.02	.47849 04164	.47770 97587	.47693 38860	.47616 27476	.47539 62933
.025	.47909 89740	.47831 73091	.47754 04351	.47676 83010	.47600 08568
.03	.47971 06415	.47892 79641	.47815 00333	.47737 69481	.47660 85085
.035	.48032 54490	.47954 17534	.47876 28604	.47798 87187	.47721 92784
.04	.48094 34268	.48015 87076	.47937 87967	.47860 36431	.47783 31965
.045	.48156 46058	.48077 88573	.47999 79231	.47922 17519	.47845 02935
.05	.48218 90174	.48140 22340	.48062 02707	.47984 30764	.47907 06006
.055	.48281 66934	.48202 88692	.48124 58712	.48046 76480	.47969 41493
.06	.48344 76659	.48265 87953	.48187 47568	.48109 54990	.48032 09716
.065	.48408 19678	.48329 20448	.48250 69600	.48172 66618	.48095 10999
.07	.48471 96323	.48392 86510	.48314 25139	.48236 11696	.48158 45674
.075	.48536 06930	.48456 86474	.48378 14522	.48299 90558	.48222 14074
.08	.48600 51843	.48521 20684	.48442 38090	.48364 03544	.48286 16540
.085	.48665 31408	.48585 89485	.48506 96188	.48428 51001	.48350 53416
.09	.48730 45978	.48650 93229	.48571 89169	.48493 33279	.48415 25052
.095	.48795 95912	.48716 32275	.48637 17389	.48558 50735	.48480 31805
.10	.48861 81574	.48782 06986	.48702 81210	.48624 03730	.48545 74035
.105	.48928 03332	.48848 17729	.48768 81001	.48689 92632	.48611 52108
.11	.48994 61563	.48914 64880	.48835 17136	.48756 17813	.48677 66399
.115	.49061 56646	.48981 48818	.48901 89994	.48822 79653	.48744 17284
.12	.49128 88969	.49048 69930	.48968 99959	.48889 78536	.48811 05147
.125	.49196 58924	.49116 28609	.49036 47425	.48957 14853	.48878 30379
.13	.49264 66912	.49184 25252	.49104 32789	.49024 89002	.48945 93376
.135	.49333 13338	.49252 60265	.49172 56456	.49093 01386	.49013 94542
.14	.49401 98614	.49321 34060	.49241 18835	.49161 52415	.49082 34285
.145	.49471 23160	.49390 47055	.49310 20345	.49230 42505	.49151 13021
.15	.49540 87400	.49459 99674	.49379 61409	.49299 72081	.49220 31173
.155	.49610 91768	.49529 92349	.49449 42459	.49369 41571	.49289 89170
.16	.49681 36704	.49600 25520	.49519 63933	.49439 51415	.49359 87449

r^2	$\frac{2}{\pi}K'$	$-\delta^{2*}$	$\frac{2}{\pi}(K'-E')$	$-\delta^{4*}$
−.005	0.99 87472 57295		−.00 25078 42901	
0	1.00	54 68680	0	156 24693
.005	1.00 12472 73896	54 19220	.00 24922 17647	154 44576
.01	24891 28446	53 70445	49689 90169	152 67443
.015	37256 12428	53 22344	74304 94709	150 93240
.02	49567 73947	52 74887	98769 05483	149 21892
.025	1.00 61826 60461	52 28085	.01 23083 93849	147 53349
.03	74033 18774	51 81911	47251 28362	145 87544
.035	86187 95061	51 36362	71272 74837	144 24424
.04	98291 34873	50 91421	95149 96405	142 63927
.045	1.01 10343 83153	50 47079	.02 18884 53572	141 06006
.05	22345 84245	50 03328	42478 04270	139 50599
.055	34297 81902	49 60156	65932 03915	137 97662
.06	46200 19297	49 17543	89248 05454	136 47138
.065	1.01 58053 39044	48 75502	.03 12427 59420	134 98977
.07	69857 83187	48 33999	35472 13982	133 53139
.075	81613 93231	47 93038	58383 14988	132 09566
.08	93322 10140	47 52604	81162 06018	130 68221
.085	1.02 04982 74349	47 12692	.04 03810 28426	129 29054
.09	16596 25771	46 73288	26329 21387	127 92024
.095	28163 03812	46 34391	48720 21939	126 57085
.10	39683 47371	45 95980	70984 65328	125 24201
.105	1.02 51157 94861	45 58064	.04 93123 83546	123 93326
.11	62586 84201	45 20616	.05 15139 08375	122 64425
.115	73970 52840	44 83639	37031 68424	121 37454
.12	85309 37756	44 47125	58802 90670	120 12380
.125	1.02 96603 75464	44 11061	.05 80454 00194	118 89166
.13	1.03 07854 02029	43 75443	.06 01986 20217	117 67773
.135	19060 53070	43 40264	23400 72138	116 48168
.14	30223 63767	43 05518	44698 75568	115 30318
.145	1.03 41343 68867	42 71193	.06 65881 48364	114 14186
.15	52421 02697	42 37284	86950 06663	112 99743
.155	63455 99167	42 03791	.07 07905 64914	111 83955
.16	74448 91771		28749 35911	

The value of the modulus k is $1/\sqrt{1+r^2}$.

$$\frac{2}{\pi}K' = \sqrt{1+r^2}\left[1-\left(\frac{1}{2}\right)^2 r^2+\left(\frac{1.3}{2.4}\right)^2 r^4-\left(\frac{1.3.5}{2.4.6}\right)^2 r^6+\cdots\right]$$

$$\frac{2}{\pi}(K'-E') = \frac{r^2}{\sqrt{1+r^2}}\left[\frac{1}{2}-\left(\frac{1}{2}\right)^2\cdot\frac{r^2}{4}+\left(\frac{1.3}{2.4}\right)^2\cdot\frac{r^4}{6}-\left(\frac{1.3.5}{2.4.6}\right)^2\cdot\frac{r^6}{8}+\cdots\right]$$

*These are modified differences; $\delta^{2*}_u = \delta^2_u - 0.184\cdot\delta^4_u$.

SOME BESSEL EQUATIONS AND THEIR APPLICATION TO GUIDE AND CAVITY THEORY*

MORRIS KLINE

1. Introduction. Information about the real, positive values of x which satisfy either of the Bessel function equations

$$J_n(x)N_n(\rho x) - J_n(\rho x)N_n(x) = 0, \qquad n = 0, 1, 2, 3 \cdots \quad (1)$$

and

$$J'_n(x)\,N'_n(\rho x) - J'_n(\rho x)\,N'_n(x) = 0, \qquad n = 0, 1, 2, 3 \cdots \quad (2)$$

for ρ values between 0 and 1, has been desirable for a number of applications in hydrodynamics, acoustics, and electromagnetic theory. For example, the existence of electromagnetic modes in coaxial, circularly cylindrical wave guides and cavities depends upon the existence of such solutions of equations (1) and (2).

Asymptotic expressions for the roots of equations (1) and (2) in terms of ρ and actual calculation of specific ones have been published. An extensive table of these roots has recently been prepared by H. B. Dwight.[1] Borgnis[2] observes on the basis of calculations that as ρ approaches 0, the first root of equation (1) for $n = 0$ approaches the first root of $J_0(x) = 0$ and likewise, the first and second roots of equation (2) for $n = 1$ approaches the first and second roots, respectively, of $J'_1(x) = 0$.

This paper establishes mathematically the existence of roots of equations (1) and (2) for ρ near 0. Moreover, it proves that the observation of Borgnis holds generally for all roots and all orders of the Bessel functions. Specifically, each solution $x(\rho)$ of equations (1) and (2) approaches, as ρ approaches zero, one and only one root of $J_n(x) = 0$ and $J'_n(x) = 0$, respectively. There is, in fact, a one-to-one correspondence between the positive roots of $J_n(x) = 0$ and solutions of equation (1) for small ρ and, likewise, a one-to-one correspondence between the positive roots of $J'_n(x) = 0$ and solutions of equation (2) for small ρ.

These facts can be applied in several ways to electromagnetic wave guide and cavity theory. In the first place the existence and ordering of coaxial, circularly cylindrical modes[3] has meaning only in terms of the existence and ordering for a given n of the solutions of equations (1) and (2). This ordering can now be established on the basis of the solutions proven to exist for small ρ. Moreover, since the circular modes correspond to the zeros[4] of $J_n(x)$ and $J'_n(x)$, there is a

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[1] Dwight, H. B.: Table of Roots for Natural Frequencies in Coaxial Type Cavities. J. Math. and Physics, 27, 84. This paper contains also references to earlier work on the roots.

[2] Borgnis, F.: Die konzentrische Leitung als Resonator, Hochfrequenz und Electroakustik, V. LVI, 1940, pp. 47–54.

[3] Hereafter coaxial, circularly cylindrical will be shortened to coaxial and circularly cylindrical to circular.

[4] We shall use the more convenient phrase zero of $J_n(x)$ for root of $J_n(x) = 0$. Likewise for $J'_n(x)$.

one-to-one correspondence between the coaxial modes other than the principal or TEM mode and the circular ones.

It may now be proved also that the field of each coaxial mode (other than the principal mode) approaches the field of the corresponding circular mode as ρ approaches 0 and that the field of a circular mode is disturbed very little by the introduction of a thin inner conductor. Thus the use of the notation $TM_{n,m}$ and $TE_{n,m}$ for coaxial modes and their corresponding circular modes is well justified.

It follows readily too that quantities such as cut-off and resonant frequencies for modes in coaxial structures with thin inner conductors may be approximated by the corresponding quantities for the corresponding modes in circular structures.

No doubt physical evidence for many of the above facts about guides and cavities has not been wanting. It does seem desirable, however, to add mathematical proof.

2. The Principal Theorem. By a solution of equation (1) [or (2)] we shall understand a function $x(\rho)$ which is defined for positive ρ values arbitrarily close to 0 and which satisfies equation (1) [or (2)] identically. Such solutions may be bounded or unbounded as ρ approaches 0. The principal theorem of this paper may now be stated thus:

Theorem. A. To each positive zero, $s_{n,m}$, of $J_n(x)$ there exists a unique, continuous solution $x(\rho)$ of the equation

$$J_n(x)\,N_n(\rho x) - J_n(\rho x)\,N_n(x) = 0, \qquad n = 0, 1, 2, \cdots, \quad (1)$$

and $x(\rho)$ approaches $s_{n,m}$ as ρ approaches 0. The same relation holds for zeros, $s'_{n,m}$, of $J_n'(x)$ and solutions of the equation

$$J'_n(x)\,N'_n(\rho x) - J'_n(\rho x)\,N'_n(x) = 0, \qquad n = 0, 1, 2, \cdots, \quad (2)$$

where primes indicate derivatives with respect to the argument.

B. As ρ approaches 0 each continuous solution of equation (1) [equation (2)] approaches a (unique) zero of $J_n(x)$ $[J'_n(x)]$.

C. The correspondence between zeros of $J_n(x)$ and solutions of equation (1) is one-to-one. Likewise the correspondence between zeros of $J'_n(x)$ and solutions of equation (2) is one-to-one.

Proof of A. The proof will be carried through for equation (1). Minor modifications necessary to handle equations (2) will be indicated later. Primarily we apply the implicit function theorem for a function of two variables to equation (1). The theorem will be applied in the neighborhood of $\rho = 0$ and $x = s_{n,m}$ but only for $\rho \geq 0$. This modification of the usual statement of the implicit function theorem is justified by the standard proofs.[5]

[5] See the proof in Courant, R.: Differential and Integral Calculus, Vol. II, Nordemann Publishing Co., N. Y., 1936, p. 119.

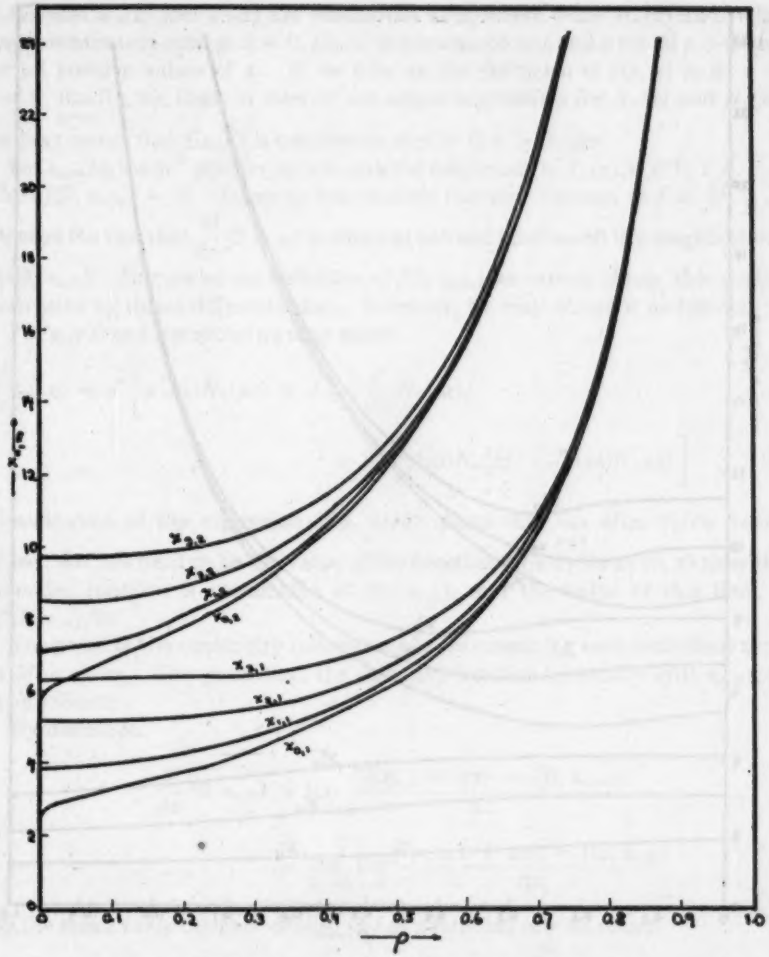

FIG. 1. Solutions of $J_n(x)\,N_n(\rho x) - J_n(\rho x)\,N_n(x) = 0$

From the expressions for $N_n(x)$, namely,

$$N_n(x) = -\sum_{r=0}^{n-1} \frac{(n-r-1)!}{n!}\left(\frac{x}{2}\right)^{-n+2r} + \sum_{r=0}^{\infty} \frac{(-1)^r}{r!}\frac{\left(\frac{x}{2}\right)^{n+2r}}{r!(n+r)!}\left\{2\log\frac{x}{2} - \psi(r+1) - \psi(n+r+1)\right\}$$

$$N_0(x) = 2\left(\gamma + \log\frac{x}{2}\right)J_0(x) - 2\sum_{r=1}^{\infty}\frac{(-1)^r\left(\frac{x}{2}\right)^{2r}}{(r!)^2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{r}\right),$$

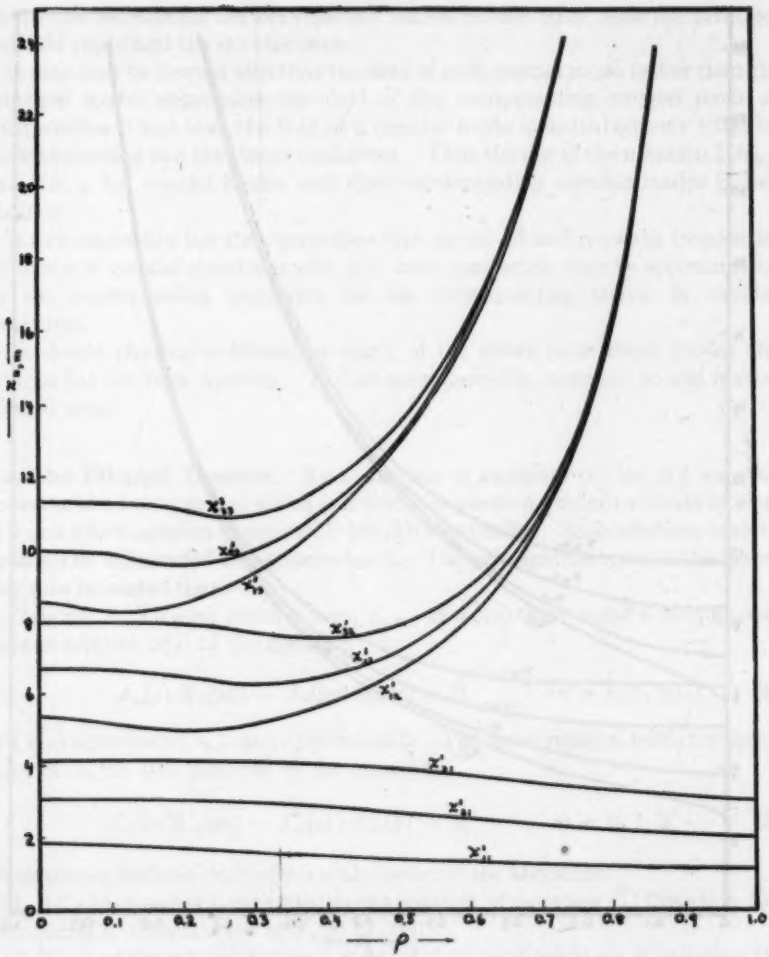

FIG. 2. Solutions of $J'_n(x)\,N'_n(\rho x) - J'_n(\rho x)\,N'_n(x) = 0$

we see that at $\rho = 0$, $N_n(\rho x)$ has a discontinuity of the order of $(\rho x)^{-n}$ and one of the order of $\log \rho x$. Hence we cannot apply the implicit function theorem directly to equation (1). However insofar as solutions of equation (1) for $\rho > 0$ are concerned we can equally well consider the equation

$$f(\rho, x) = \begin{cases} \rho^n[J_n(x)N_n(\rho x) - J_n(\rho x)N_n(x)] = 0, & \text{for } n = 1, 2, 3 \cdots \\ & \text{in equation (1)} \\ \dfrac{1}{\log \rho}\Big[J_0(x)N_0(\rho x) - J_0(\rho x)N_0(x)\Big] = 0, & \text{for } n = 0 \text{ in equation (1).} \end{cases}$$

Because $J_n(x)$ and $J_0(x)$ are continuous everywhere while $N_n(x)$ and $N_0(x)$ are discontinuous only at $x = 0$, $f(\rho, x)$ is continuous in ρ and x for all $\rho > 0$ and for all positive values of x. If we take as the definition of $f(\rho, x)$ at $\rho = 0$, $x \neq 0$, $\lim_{\rho \to 0} f(\rho, x)$, then, in view of the actual expressions for $N_n(x)$ and $N_0(x)$, we may assert that $f(\rho, x)$ is continuous at $\rho = 0$, $x > 0$, also.

Let $s_{n,m}$ be the m^{th} positive zero in order of magnitude of $J_n(x)$, $n = 0, 1, 2, \cdots$. Then $f(0, s_{n,m}) = 0$. To apply the implicit function theorem to f at $(0, s_{n,m})$ we need the fact that $\frac{\partial f}{\partial x}(0, s_{n,m})$ be unequal to 0 and continuous in a neighborhood of $(0, s_{n,m})$. In view of the definition of $f(0, s_{n,m})$ we cannot obtain this partial derivative by direct differentiation. However, we may obtain it as follows.

For $\rho > 0$ and x positive we may write

$$\frac{\partial f}{\partial x}(\rho, x) = \rho^n \left[J_n'(x) N_n(\rho x) + J_n(x) \frac{\partial}{\partial x} N_n(\rho x) - \frac{\partial}{\partial x} J_n(\rho x) N_n(x) - J_n(\rho x) N_n'(x) \right]. \tag{3}$$

Examination of the expression $\partial f(\rho, x)/\partial x$ shows that $\lim_{\rho \to 0} \partial f(\rho, x)/\partial x$ exists. If we take this limit to be the value of the function $\partial f(\rho, x)/\partial x$ at $(0, x)$ then the extended function is continuous at $(0, s_{n,m})$, and the value of this limit is $\partial f(0, s_{n,m})/\partial x$.

The proof of the continuity follows merely by examining each individual term in $\partial f(\rho, x)/\partial x$. The proof that the extended function is actually $\partial f(0, s_{n,m})/\partial x$ is as follows:

By definition,

$$\frac{\partial f}{\partial x}(0, s_{n,m}) = \lim_{\Delta x \to 0} \frac{f(0, s_{n,m} + \Delta x) - f(0, s_{n,m})}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \lim_{\rho \to 0} \frac{f(\rho, s_{n,m} + \Delta x) - f(\rho, s_{n,m})}{\Delta x}$$

By the mean value theorem applied to f as a function of x we obtain

$$= \lim_{\Delta x \to 0} \lim_{\rho \to 0} \frac{\partial f}{\partial x}(\rho, x_1), \qquad s_{n,m} < x_1 < s_{n,m} + \Delta x$$

In view of the continuity at $(0, s_{n,m})$ of the function denoted by $\partial f(\rho, x_1)/\partial x$ we may write

$$= \lim_{\rho \to 0} \lim_{\Delta x \to 0} \frac{\partial f}{\partial x}(\rho, x_1) = \lim_{\rho \to 0} \frac{\partial f}{\partial x}(\rho, s_{n,m}).$$

The value of $\partial f(0, s_{n,m})/\partial x$ may be obtained, then, from equation (3) by letting ρ approach 0 and by using the fact that $J_n(s_{n,m}) = 0$. All terms but $\rho^n J_n'(x) N_n(\rho x)$ yield 0 in the limit. The limit of this term is

$$J'_n(s_{n,m})\ [-(n-1)!\ (2/s_{n,m})^n].$$

Now $J'_n(x)$ cannot vanish at the zeros of $J_n(x)$, for $J'_n(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$ and the positive zeros of $J_n(x)$ and $J_{n+1}(x)$ separate each other. If $J'_n(s_{n,m})$ and $J_n(s_{n,m})$ were both zero we would have a contradiction. Hence the term in question is not zero and we have that $\partial f(0, s_{n,m})/\partial x \neq 0$.

By the usual implicit function theorem we may now assert the existence for ρ near 0 of a unique continuous function $x(\rho)$ which satisfies equation (1) and such that $x(0) = s_{n,m}$. It follows that $\lim_{\rho \to 0} x(\rho) = s_{n,m}$.

The proof of the analogous assertion relating solutions of equation (2) to zeros $s'_{n,m}$ of $J'_n(x)$ proceeds in practically the same manner. Since $N'_n(x)$ is of the same general character as $N_n(x)$, except that the discontinuity at $x = 0$ of order x^{-n} is replaced by one of order x^{-n-1}, we multiply the left side of equation (2) by ρ^{n+1} and proceed as above for $n = 1, 2, 3 \cdots$. The case $n = 0$ is identical with the case $n = 1$ of equation (1) and so is already disposed of.

In paralleling the proof used for equation (1), one needs the fact that $J''_n(x)$ does not vanish where $J'_n(x)$ does. This follows from the differential equation which J_n satisfies, namely,

$$J''_n(x) + \frac{1}{x} J'_n(x) + \left(1 - \frac{n^2}{x^2}\right) J_n(x) = 0.$$

If x_1 is a zero of $J''_n(x)$ and $J'_n(x)$ then either $J_n(x_1) = 0$ which was shown above to be impossible or $x_1 = \pm n$. The latter is also impossible since the smallest positive root of $J'_n(x) > n$.[6]

We have therefore, thus far, to each positive zero of $J_n(x)$ and of $J'_n(x)$ the existence of a unique continuous function $x(\rho)$ in the interval $0 \leq \rho < c$, where c is some positive number; $x(\rho)$ satisfies equation (1), resp. (2); and $\lim_{\rho \to 0} x(\rho) = s_{n,m}$, resp., $s'_{n,m}$.

Proof of B. We must now show that each continuous solution of equations (1) and (2) approaches a zero of $J_n(x)$ and $J'_n(x)$, respectively, as ρ approaches 0. We shall however consider more general solutions of equations (1) and (2) and thereby establish somewhat more than the theorem requires.

Let $x(\rho)$ be any solution of equation (1) [or (2)] which is defined for ρ values arbitrarily close to 0. We shall show that

1) $x(\rho)$ cannot have 0 as a limiting value as ρ approaches 0.

2) Any finite limiting value approached by $x(\rho)$ must be a zero of $J_n(x)$ [or of $J'_n(x)$].

3) Any solution $x(\rho)$ which is unbounded for ρ near 0 must be discontinuous.

As in the first part of this proof we shall consider equation (1) and then indicate the necessary modifications to handle equation (2).

Proof of 1). Suppose that $x(\rho)$, a solution of equation (1), has 0 as a lim-

[6] See Watson, G. N.: A Treatise on the Theory of Bessel Functions, Macmillan Co., N. Y., 1945, p. 485.

iting value as ρ approaches 0. This assumption will be shown to lead to a contradiction. We rewrite equation (1) in the form

$$[J_n(x)/J_n(\rho x)] - [N_n(x)/N_n(\rho x)] = 0. \tag{4}$$

and consider first the cases $n > 0$.

Equation (4) can then be written in the form:

$$\frac{\frac{1}{n}\left(\frac{x}{2}\right)^n + R_1(x)}{\frac{1}{n!}\left(\frac{\rho x}{2}\right)^n + R_1(\rho x)} - \frac{C(x)\log\frac{x}{2} + D(x) - \sum_{r=0}^{n-1}\frac{(n-r-1)!}{r!}\left(\frac{x}{2}\right)^{-n+2r}}{C(\rho x)\log\frac{\rho x}{2} + D(x) - \sum_{r=0}^{n-1}\frac{(n-r-1)!}{r!}\left(\frac{\rho x}{2}\right)^{-n+2r}} = 0. \tag{5}$$

where $R_1(x)$ is a power series beginning with a term in x^{n+2} and $C(x)$ and $D(x)$ are power series each beginning with a term in x^n. In the first fraction we divide numerator and denominator by x^n, and in the second fraction we multiply numerator and denominator by $(\rho x)^n$. There results

$$\frac{\frac{1}{n!}\left(\frac{1}{2}\right)^n + R_2(x)}{\frac{1}{n!}\left(\frac{\rho}{2}\right)^n + R_2(\rho x)} - \frac{(\rho x)^n C(x)\log\frac{x}{2} + (\rho x)^n D(x) - \rho^n\sum_{r=0}^{n-1}\frac{(n-r+1)!}{r!}\left(\frac{1}{2}\right)^{-n+2r}x^{2r}}{(\rho x)^n C(x)\log\frac{\rho x}{2} + (\rho x)^n D(\rho x) - \sum_{r=0}^{n-1}\frac{(n-r+1)!}{r!}\left(\frac{1}{2}\right)^{-n+2r}(\rho x)^{2r}} = 0, \tag{6}$$

where $R_2(x)$ and $R_2(\rho x)$ result from dividing $R_1(x)$ and $R_1(\rho x)$ by x^n, and therefore have x^2 as the lowest power of x. If we now consider the value of the left-hand side of equation (6) when $x(\rho)$ is substituted for x and ρ approaches 0, we find that the first fraction becomes infinite while the second fraction approaches 0, for x-values near 0. The whole expression, however, should be identically 0 for all $x(\rho)$ being considered. In view of the contradiction x cannot have 0 as a limiting value as ρ tends to 0.

To treat the case in which $n = 0$, equation (4) can be written as

$$\frac{J_0(x)}{J_0(\rho x)} - \frac{(\nu - \log 2 + \log x)J_0(x) - D(x)}{(\nu - \log 2 + \log \rho x)J_0(\rho x) - D(\rho x)} = 0, \tag{7}$$

where ν is a constant and $D(x)$ is a power series beginning with a term in x^2. We first rewrite equation (7) in the form

$$\frac{J_0(x)}{J_0(\rho x)}\left[1 - \frac{\nu - \log 2 + \log x - [D(x)/J_0(x)]}{\nu - \log 2 + \log \rho x - [D(\rho x)/J_0(\rho x)]}\right] = 0.$$

For x and ρx near 0, neither $J_0(x)$ nor $J_0(\rho x)$ can be 0, since $J_0(x)$ approaches 1 as x approaches 0. Hence, for the values of $x(\rho)$ that have been assumed to satisfy equation (1),

$$1 - \frac{\nu - \log 2 + \log x - [D(x)/J_0(x)]}{\nu - \log 2 + \log \rho x - [D(\rho x)/J_0(\rho x)]} \equiv 0. \tag{8}$$

Simplifying (8) gives

$$\frac{\log \rho - [D(\rho x)/J_0(\rho x)] + [D(x)/J_0(x)]}{\nu - \log 2 + \log \rho x - [D(\rho x)/J_0(\rho x)]} \equiv 0. \tag{9}$$

For equation (9) to hold identically, either the numerator must be zero or the denominator infinite. However, for values of ρ, x, and ρx near 0, the first term of the numerator will be large while the second and third terms will be near 0. Hence the numerator cannot be zero. The denominator is dominated by the term, $\log \rho x$, and this term is infinite only when $x = 0$. For $\rho \neq 0$ and $x = 0$ equation (1) is meaningless, and hence such values need not be considered. We can therefore exclude the possibility of the denominator being infinite.

Since the assumption that $x(\rho)$ tends to 0 with ρ leads to a contradiction we conclude that $x(\rho)$ cannot have 0 as a limiting value.

Proof of 2). We now consider a finite limiting value of $x(\rho)$ as ρ approaches 0 and prove that it must be a zero of $J_n(x)$. In this case ρx also tends to 0 as ρ does. We consider equation (1) in the form

$$J_n(x) - [N_n(x)/N_n(\rho x)]\, J_n(\rho x) = 0. \tag{10}$$

wherein x stands for $x(\rho)$. We first note that, as a consequence of part 1), $x(\rho)$ cannot have 0 as a limiting value as ρ approaches 0. For any other finite limiting value the second term in equation (10) approaches 0 with ρ. Hence the corresponding values of $J_n(x)$ must approach 0, and since J_n is a continuous function of x, the limiting value approached by x must be a zero of $J_n(x)$.

Since $x(\rho)$ may be a discontinuous function of ρ there can be many limiting values approached by $x(\rho)$ as ρ tends to 0. But each must be a zero of $J_n(x)$. Consider the values of $x(\rho)$ which approach one particular zero, $s_{n,m}$, of $J_n(x)$. For ρ sufficiently near 0 these values all lie in an arbitrarily small interval of x-values about $s_{n,m}$. Now the implicit function theorem asserts that any solution $x(\rho)$ of equation (1) in the neighborhood of $(0, s_{n,m})$ is unique in a sufficiently small neighborhood of that point. It therefore follows that the values of $x(\rho)$ which approach any one $s_{n,m}$ must for ρ sufficiently small belong to the continuous solution already shown in part A to exist and to approach $s_{n,m}$.

Proof of 3). We have finally to consider the case of an unbounded solution, $x(\rho)$, of equation (1) and to prove that such a solution must be discontinuous in any neighborhood of $\rho = 0$.

As x becomes infinite, ρx may or may not become infinite. We consider each possibility separately, and first suppose that ρx tends to infinity as ρ tends to 0. For large values of x we may use asymptotic expansions for $J_n(x)$ and $N_n(x)$,[7] and write

[7] The statements about the asymptotic expansions for these functions that are used here are established in Watson, G. N., loc. cit., p. 199.

$$J_n(x) - \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right) = \epsilon_1(x),$$
$$N_n(x) - \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right) = \epsilon_2(x), \tag{11}$$

where $\epsilon_1(x)$ and $\epsilon_2(x)$ are defined by the left-hand sides of equations (11), and both are of the order of $1/x^{3/2}$.

If we now substitute from equations (11) into equation (1), collect the principal terms, and apply a trigonometric identity, we obtain

$$(1/x\sqrt{\rho}) \sin(\rho x - x) + \epsilon_3(x, \rho x) = 0, \tag{12}$$

where the lowest order term in $\epsilon_3(x, \rho x)$ is that in $1/x^2\rho^{3/2}$.

The explicit solutions of equation (1) may be obtained from equation (12) which is its exact equivalent. Multiplying through by $x\sqrt{\rho}$, we have

$$\sin(\rho x - x) + \epsilon_4(x, \rho x) = 0,$$

where the order of $\epsilon_4(x, \rho x)$ is that of $1/\rho x$. Hence $\rho x - x = m\pi + \arcsin \epsilon_4(x, \rho x)$, $m = 0, \pm 1, \pm 2, \cdots$

and
$$x = \frac{m\pi}{\rho - 1} + \frac{\arcsin \epsilon_4(x, \rho x)}{\rho - 1}. \tag{13}$$

Now as ρ approaches 0 and x and ρx, in accordance with our assumption, become infinite, ϵ_4 must tend to 0. The continuous solutions of equation (13), and therefore of equation (1), must approach finite quantities. Hence a discontinuous function, $x(\rho)$, satisfying equation (1) and tending to infinity as ρ approaches 0, can be constructed only by varying the value of m through integral steps as ρ tends to 0.

We now consider the case in which ρx remains bounded as x becomes infinite for some sequence of ρ values approaching 0. The values of ρx must then have a finite limiting value, which we shall call L.

Let us write equation (1) in the form

$$[J_n(x)/N_n(x)] - [J_n/(\rho x)/N_n(\rho x)] = 0. \tag{14}$$

The fraction $[J_n(\rho x)/N_n(\rho x)]$ is a continuous function of ρx for $\rho x \neq 0$, and by defining $J_n(0)/N_n(0)$ to be 0, this fraction is made continuous for $\rho x = 0$ as well. Now let ρ take on a sequence of values tending to 0, for which x becomes infinite and ρx approaches L. Then for any positive ϵ, ρ can be made sufficiently small so that $J_n(\rho x)/N_n(\rho x)$ differs from $J_n(L)/N_n(L)$ by less than ϵ. The fraction $J_n(x)/N_n(x)$, in view of the asymptotic expression given in equation (11), behaves more and more like $\cot\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right)$ as x becomes infinite. Now if $x(\rho)$ were a continuous function of ρ, as ρ tends to 0 the values of the above fraction would assume all values from $-\infty$ to $+\infty$. Consequently equation (14) could not be satisfied identically.

The proof of part B) is now complete for equation (1). The proof in the case of equation (2) is practically the same. We note that J_0' and N_0' are equal to

J_1 and N_1, respectively, that J'_n and N'_n have the same general properties as J_n and N_n and that asymptotic expansions for J'_n and N'_n can be obtained by using the recurrence relation

$$J'_n(x) = \tfrac{1}{2}[J_{n-1}(x) - J_{n+1}(x)] \tag{15}$$

and a similar one for $N'_n(x)$, together with the theorem that permits the addition of the asymptotic expansions of two functions in order to obtain an asymptotic expansion of the sum.

Corollary. For ρ sufficiently small the values of any discontinuous solutions of equation (1) or (2) must belong to one or more of the continuous solutions.

3. Application of the Principal Theorem to Electromagnetic Guide and Cavity Theory. It is the custom to identity the electromagnetic field configurations or modes in circular guides by the indices n, m. The n refers to the order of J_n and of J'_n which are involved in the field expressions for that mode; the m denotes the order, in terms of magnitude, of the positive zero of J_n or of J'_n which is involved in the field expressions and in other formulas belonging to that mode. Cavity resonant modes are denoted by three indices, n, m, and l, the n and m having the same meaning as for guides and l denoting the number of half-sinusoidal variations in the field along the length of the guide. In practice the symbols $TE_{n,m}$ and $TM_{n,m}$ are used for guides and $TE_{n,m,l}$ and $TM_{n,m,l}$ are used for cavities, the TE or TM depending upon whether the zeros of J'_n or of J_n are involved. A similar system of indices and symbols is used for coaxial structures. The n refers to the order of J_n, N_n, J'_n and N'_n, all of which are involved in the field expressions of any one guide or cavity mode, and the l has the same meaning as above. For a given ratio, ρ of the radii, the value $x(\rho)$ satisfying equation (1) or (2) takes the place which the zero of J_n or of J_n' occupies in the theory of circular structures.[8]

Our principal theorem bears directly on the theory of coaxial structures. For a given ρ as many modes exist as do solutions of equations (1) or (2) and these modes may be ordered, insofar as the second index m is concerned, in accordance with the values of $x(\rho)$. Moreover, since the principal theorem establishes a one-to-one correspondence between the zeros of J_n and of J'_n on the one hand and solutions of equations (1) and (2) on the other, the m^{th} zero corresponding to the m^{th} $x(\rho)$, it follows that there are "as many" coaxial modes as circular modes and that the coaxial modes remain distinct as ρ varies near zero.

The principal theorem can now be used to prove a more significant result.

THEOREM. The $TE_{n,m}$ $[TM_{n,m}]$ coaxial guide mode approaches the $TE_{n,m}$ $[TM_{n,m}]$ circular guide mode as ρ the ratio of the radii of the coaxial conductors approaches zero. The corresponding statement applies also to cavities. Moreover, each circular guide and cavity mode is so approached.[9]

The proof of this theorem follows immediately by applying the principal

[8] Some of the above facts are presented in greater detail in Sarbacher, R. I. and Edson, Wm. A.: Hyper- and Ultra-high Frequency Engineering, John Wiley & Sons, N. Y., 1943, Chap. 7.

[9] The principal coaxial mode, the *TEM* mode, does not appear in this correspondence.

theorem to the field expressions for circular and coaxial guide and cavity modes.[10] A typical expression for one of the six field components in a $TM_{n,m}$ coaxial guide mode is

$$E_z = A \cos n\phi \left[J_n\left(r\frac{x_{n,m}}{b}\right) - \alpha N_n\left(r\frac{x_{n,m}}{b}\right)\right] e^{j(\omega t - \beta_{n,m} z)} \tag{16}$$

where A is merely an amplitude constant; r, ϕ, & x are cylindrical coordinates; a and b are the radii of the inner and outer coaxial conductors; $x_{n,m}$ is the value of the m^{th} solution of equation (1) for a given ρ, the ratio of the radii; $\beta_{n,m} = \sqrt{\omega^2\epsilon\mu - (x_{n,m}/b)^2}$; ω is the circular frequency; and

$$\alpha = J_n[(a/b)x_{n,m}]/N_n[(a/b)x_{n,m}].$$

We now fix on a particular position in a coaxial guide, that is, we fix r, ϕ, and x, and allow a, the radius of the inner conductor, to approach zero. Since $x_{n,m}$ approaches a definite, positive zero of $J_n(x)$ as $\frac{a}{b}$ approaches zero, α approaches 0. Moreover, since N_n in the expressiou for E_z will have a non-zero argument in the limit, the term αN_n also approaches zero. Terms involving $x_{n,m}$ will approach the same expressions with $s_{n,m}$ instead of $x_{n,m}$. The entire expression for E_z will then approach precisely the E_z of a circular guide supporting the $TM_{n,m}$ mode.

This argument applies to all the field components, in all the modes (except the TEM or principal coaxial mode) of both guides and cavities. In view of the one-to-one correspondence between zeros of $J_n(x)$ and $J_n'(x)$ and solutions of equations (1) and (2), respectively, it follows also that each circular guide and cavity mode is approached by a coaxial one.[11]

The conclusion that each circular mode is approached by a coaxial mode is somewhat surprising in the case of a circular mode with a longitudinal component of electric field at the center, for there can be no longitudinal electric field along the surface of the inner coaxial conductor. Actually the introduction of an inner conductor in a circular guide or cavity disturbs the field of the guide only in a very small neighborhood of the inner conductor and this neighborhood becomes smaller as the radius of the inner conductor decreases.[12] Quantitatively the effect of the inner conductor is contained in the term

[10] The explicit field expressions for circular guides are found in Sarbacher and Edson, loc. cit., pp. 248 and 258. Field expressions for circular cavities are in Borgnis, F.: Electromagnetic Waves in Dielectric Spaces, Annalen der Physik, V. 35, 1939, pp. 359–384. The field expressions for coaxial guides and cavities can be derived in the same manner as in Sarbacher and Edson and Borgnis. The only essential difference is the use of both kinds of Bessel Functions in the coaxial case.

[11] This theorem has bearing on the matter of standardizing notation for the circularly cylindrical modes. See the article by Kirkman, R. and Kline, M.: The Transverse Electric Modes in Coaxial Cavities, Proc. I. R. E., V. 34, 1946, pp. 14–17, and discussion on this paper in V. 35, 1947, pp. 931–935.

[12] This corresponding phenomenon in sound was noted by Rayleigh, Lord: Theory of Sound, Dover Publications, N. Y., 1945, V. I, p. 350 and is briefly discussed by Borgnis, loc. cit., p. 50.

$$aN_n\left(r\frac{x_{n,m}}{b}\right) = \left[J_n\left(\frac{a}{b}x_{n,m}\right) \Big/ N_n\left(\frac{a}{b}x_{n,m}\right)\right] N_n\left(r\frac{x_{n,m}}{b}\right)$$

of expression (16). While for a fixed r value, the term approaches zero as the inner radius, a, approaches zero, for a fixed a and b the factor $N_n(rx_{n,m}/b)$ becomes large as r approaches 0. In more mathematical form one may state that the approach of E_z in expression (16) to the E_z of the circular guide is non uniform in r as a approaches 0.

Further application of the principal theorem may be made to the calculation of cut-off frequencies of a coaxial guide, resonant frequencies of a coaxial cavity, dimensions of a coaxial guide or cavity for a given frequency and a given mode, wave length in a guide or cavity, etc. All these quantities may be approximated in a coaxial guide or cavity with small inner conductor by the corresponding quantity in a circular guide or cavity, for the formulas for these quantities are formally the same in coaxial and in circular structures with $x_{n,m}$ replacing $s_{n,m}$ and $x'_{n,m}$, the m^{th} value of $x(\rho)$ satisfying equation (2), replacing $s'_{n,m}$. The precise $x_{n,m}$ values, on the other hand, are quite lengthy to compute.[13]

Some graphs of the variation of the first few roots of equations (1) and (2) with ρ, the ratio of the radii are offered herewith. They not only illustrate the principal theorem of the paper but may be used for approximate values of $x_{n,m}$ and $x'_{n,m}$.

One incidental point connected with carrying out the calculation of the graphs may be worth noting. Since Bessel function tables are not very extensive the following device can be used. Equation (2) can be written thus:

$$J'_n(x) = N'_n(x) \cdot [J'_n(\rho x)/N'_n(\rho x)]. \tag{2}$$

Let $s_{n,p}$ and $s_{n,p+1}$ be two successive zeros of $J'_n(x)$. When $\rho = s_{n,p}/s_{n,p+1}$ and $x = s_{n,p+1}$, equation (2) is satisfied. Hence, for this value of ρ, $s_{n,p+1}$ is a root of (2) as well as of $J'_n(x)$. By letting p take on the values 1, 2, $\cdots$ the calculation of the second solution[14] of equation (2) for many ρ values falls back on a knowledge of the zeros of $J_n'(x)$ itself. By using the ratio $s_{n,p}/s_{n,p+2}$ and letting p take on the values of 1, 2, 3 $\cdots$ we obtain some values of the third solution of (2) in terms of ρ; etc.

A rather significant conclusion, not readily predictable theoretically, is evident from these graphs and the formula

$$f_c = cx'_{n,m}/2\pi b$$

which gives the cut-off frequency of the $TE_{n,m}$ modes in coaxial guides (c is the velocity of light in the dielectric of the guide and b, the outer radius). For the coaxial $TE_{n,1}$ modes only, the cut-off frequencies decrease as a increases while b is held fixed; that is, longer wave-lengths can be accommodated. Likewise the resonant frequency of the $TE_{n,1,l}$ coaxial cavity modes will decrease as a increases for fixed b.

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[13] Many of these formulas are given in Sarbacher and Edson, loc. cit.

[14] The fact that it is the second solution follows from graphical considerations.

NOTE ON THE LIFT OF A TRIANGULAR WING AT SUPERSONIC SPEEDS*

By Richard Calvin Roberts

This note is concerned with finding the lift on a wing with a triangular plane form at supersonic speeds. H. J. Stewart[1] has found the lift on a delta wing lying entirely within the Mach cone. The delta wing considered by him has an isosoceles triangular plan form with the base of the triangle perpendicular to the free stream flow. The generalized delta wing considered in this paper has any triangular plan form whose base is perpendicular to the free stream flow. After the work had been done,* the author is subsequently informed that results similar to these have been obtained by P. A. Lagerstrom and W. D. Hayes.

The linearized theory of supersonic flow is based on the differential equation

$$(1 - M^2)\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2 = 0 \tag{1}$$

where (x,y,z) are rectangular cartesian coordinates, M is the Mach number of the free stream, and f is, say, one of the velocity components (u,v,w). By introducing the transformation

$$\begin{cases} r = [x^2/(M^2 - 1) - (y^2 + z^2)]^{\frac{1}{2}} = R\cos\omega/\mu\sqrt{M^2 - 1}, \\ \mu = [1 - (M^2 - 1)(y^2 + z^2)/x^2]^{\frac{1}{2}} = [1 - (M^2 - 1)\tan^2\omega]^{-\frac{1}{2}}, \\ \theta = \tan^{-1}(y/z), \end{cases} \tag{2}$$

where $R = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, and considering the case where f is independent of r, the equation (1) transforms into

$$(\mu^2 - 1)\frac{\partial}{\partial\mu}\left[(\mu^2 - 1)\frac{\partial f}{\partial\mu}\right] + \frac{\partial^2 f}{\partial\theta^2} \quad 0. \tag{3}$$

Then if

$$s = \sqrt{(\mu - 1)/(\mu + 1)} \tag{4}$$

equation (3) becomes

$$s\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right) + \frac{\partial^2 f}{\partial\theta^2} = 0 \tag{5}$$

which is the normal form of the Laplace equation in two dimensional polar coordinates. The relation between s and ω is

$$\sqrt{M^2 - 1}\tan\omega = 2s/(1 + s^2). \tag{6}$$

* An abstract of this paper appeared in the November 1946 issue of the Bulletin of the American Mathematical Society, vol. 52, p. 1012. The material in this paper is taken from a Master's thesis at Brown University prepared during the summer of 1946 under the direction of Prof. C. C. Lin.

[1] Quarterly of Applied Mathematics 4, Oct. 1946, p. 246.

We see also that the points $s = 1$ are on the Mach cone through the origin.

The quantities represented by f can be written as the real (or imaginary) parts of an analytic function of the complex variable ζ, where

$$\zeta = sl^{i\theta}. \tag{7}$$

Thus, the function

$$f + i\bar{f} = F(\zeta) \tag{8}$$

may represent any of the functions

$$u + i\bar{u} = U(\zeta), \quad v + i\bar{v} = V(\zeta), \quad w + i\bar{w} = W(\zeta). \tag{9}$$

Using the vorticity relations

$$\frac{\partial v}{\partial z} = \frac{\partial w}{\partial y}, \tag{10}$$

$$\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \tag{11}$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \tag{12}$$

the relation

$$\frac{dU}{d\zeta} = -\frac{2\zeta}{\sqrt{M^2 - 1}\,(1 + \zeta^2)}\,\frac{dW}{d\zeta} \tag{13}$$

can be derived.

The problem of finding the lift on a delta wing for small angles of attack is thus reduced to a boundary value problem in two dimensions. The z axis is taken normal to the airfoil. The conditions in the ζ plane are shown in Fig. 1.

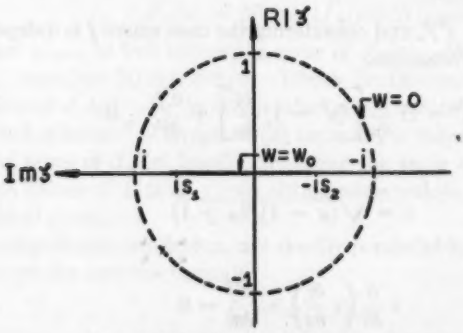

FIG. 1. Boundary conditions in the ζ plane

To the first approximation, the airfoil lies in the plane $z = 0$, and therefore it cuts the ζ plane on the imaginary axis where $\cos\theta = 0$. The boundary conditions on w are

$$w = 0 \text{ on } s = 1, \; w = -U\alpha \text{ on the airfoil,} \tag{14}$$

where α is the angle of attack. Since $s = 1$ corresponds to the Mach cone through the origin (cf. (2) and (4)) and since the flow is undisturbed outside this

cone, w must vanish on the circle $s = 1$. Also, on the airfoil the component of the stream velocity in the direction of the negative z axis is given approximately by $-U\alpha$ for small α.

Stewart has solved the problem when $s_1 = s_2$. It is the purpose of this paper to generalize this to the case when $s_1 \neq s_2$.

This may be done by a series of conformal transformations.

First, apply the transformation

$$\zeta_1 = -\frac{i}{2}\left(\zeta - \frac{1}{\zeta}\right). \tag{15}$$

This maps the interior of the unit circle into the entire ζ_1 plane, with $R1\zeta > 0$ corresponding to $Im\zeta_1 > 0$. The transformed plane is shown in Fig. 2.

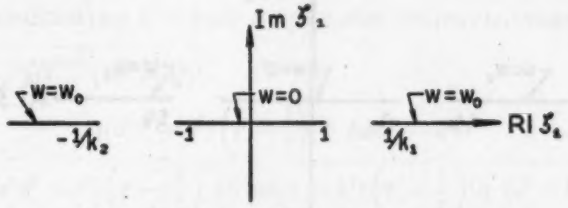

FIG. 2. Boundary conditions in the ζ_1 plane

The points at the wing tips $\zeta = is_1$ and $\zeta = -is_2$ transform into the points $\zeta_1 = 1/k_1$ and $\zeta_1 = -1/k_2$ respectively, where

$$\begin{aligned} k_1 &= 2s_1/(1 + s_1^2) = \sqrt{M^2 - 1}\tan\omega_1 \\ k_2 &= 2s_2/(1 + s_2^2) = \sqrt{M^2 - 1}\tan\omega_2 . \end{aligned} \tag{16}$$

Second, apply the transformation

$$\zeta_1 = (a\zeta_2 + b)/(b\zeta_2 + a), \tag{17}$$

where

$$a = k_1 - k_2 , b = 1 - k_1k_2 - \sqrt{(1 - k_1^2)(1 - k_2^2)}. \tag{18}$$

This transforms $Im\zeta_1 \gtreqless 0$ into $Im\zeta_2 \gtreqless 0$ while the points $\zeta_1 = 1/k_1$ and $\zeta_1 = -1/k_2$ transform into the points $\zeta_2 = 1/k$ and $\zeta_2 = -1/k$ respectively. The relation between k, k_1, and k_2 is

$$(k_1 + k_2)k = 1 + k_1k_2 - \sqrt{(1 - k_1^2)(1 - k_2^2)} \tag{19}$$

The ζ_2 plane is shown in Fig. 3.

The problem has now been reduced to the case considered by Stewart. The transformation

$$\zeta_3 = \int_0^{\zeta_2} \frac{d\zeta_2}{\sqrt{(1 - \zeta_2^2)(1 - k^2\zeta_2^2)}} \tag{20}$$

may now be applied. This defines ζ_3 as an elliptic function of ζ_2 with the modulus k. Then the region $Im\zeta_2 > 0$ is mapped into the rectangle having its corners at $\zeta_3 = \pm K, iK' \pm K$, where K and K' are the complete elliptic integrals of the first kind having moduli of k and k', where

$$k' = \sqrt{1 - k^2}. \tag{21}$$

The complete ζ_2 plane is transformed into the ζ_3 plane in the manner shown in Fig. 4.

The function $dW/d\zeta_3$ must be doubly periodic in the ζ_3 plane with periods $4K$ and $2iK'$; and, as Stewart has shown, is represented by the Jacobian elliptic function

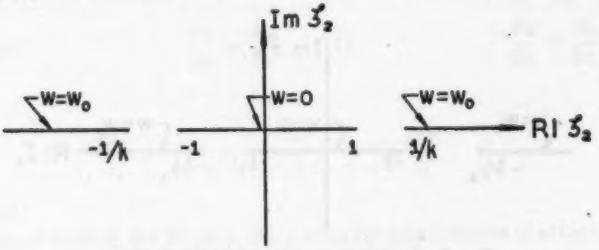

FIG. 3. Boundary conditions in the ζ_2 plane

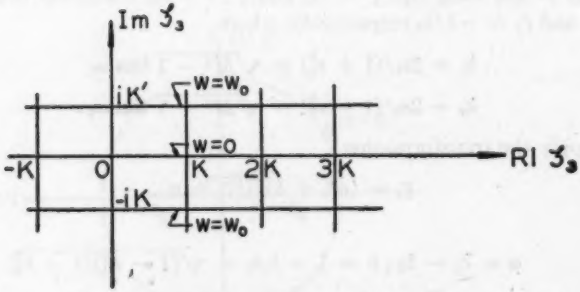

FIG. 4. Boundary conditions in the ζ_3 plane

$$dW/d\zeta_3 = iDcd^2(\zeta_3), \tag{22}$$

where

$$D = -k^2 w_0/E(k'). \tag{23}$$

In (23), $E(k')$ is the complete elliptic integral of the second kind.

If ζ_3 is eliminated from equation (22) by using the identity

$$\frac{dW}{d\zeta} = \frac{dW}{d\zeta_3} \cdot \frac{d\zeta_3}{d\zeta_2} \cdot \frac{d\zeta_2}{d\zeta_1} \cdot \frac{d\zeta_1}{d\zeta} \tag{24}$$

and referring to (15), (17), and (20), one obtains

$$\frac{dW}{d\zeta} = -\frac{k^2 w_0}{2E(k')} \cdot \frac{\frac{1}{2}(a^2-b^2)^{\frac{1}{2}}(1+\zeta^2)^2}{\zeta^3\left[\frac{1}{4}(a^2k^2-b^2)\left(\zeta-\frac{1}{\zeta}\right)^2 + iab(1-k^2)\left(\zeta-\frac{1}{\zeta}\right) + (a^2-b^2k^2)\right]^{\frac{1}{2}}} \tag{25}$$

Then from equation (13)

$$\frac{dU}{d\zeta} = \frac{k^2 w_0}{2E(k')\sqrt{M^2-1}} \cdot \frac{(a^2-b^2)^{\frac{1}{2}}(1+\zeta^2)}{\zeta^2\left[\frac{1}{4}(a^2k^2-b^2)\left(-\frac{1}{\zeta}\right)^2 + iab(1-k^2)\left(\zeta-\frac{1}{\zeta}\right) + (a^2-b^2k^2)\right]^{\frac{1}{2}}} \tag{26}$$

Using the condition that $U = 0$ at $\zeta = 1$, equation (25) may be integrated to give

$$U = \frac{-w_0}{E(k')\sqrt{M^2-1}\,(a^2-b^2)^{\frac{1}{2}}} \cdot \left\{ \frac{\frac{1}{2}(a^2k^2-b^2)\left(\zeta-\frac{1}{\zeta}\right) + iab(1-k^2)}{\left[\frac{1}{4}(a^2k^2-b^2)\left(\zeta-\frac{1}{\zeta}\right)^2 + iab(1-k^2)\left(\zeta-\frac{1}{\zeta}\right) + (a^2-b^2k^2)\right]^{\frac{1}{2}}} - \frac{iab(1-k^2)}{(a^2-b^2k^2)^{\frac{1}{2}}} \right\}. \tag{27}$$

On the top and bottom sides of the airfoil $\zeta = i\eta$, where $-\varepsilon_2 < \eta < \varepsilon_1$. Only the first term in the braces gives rise to a real term, so that

$$u = \frac{-w_0}{E(k')\sqrt{M^2-1}\,(a^2-b^2)^{\frac{1}{2}}} \cdot \frac{\frac{1}{2}(a^2k^2-b^2)\left(\eta+\frac{1}{\eta}\right) + ab(1-k^2)}{\left[\frac{1}{4}(a^2k^2-b^2)\left(\eta+\frac{1}{\eta}\right)^2 + ab(1-k^2)\left(\eta+\frac{1}{\eta}\right) - (a^2-b^2k^2)\right]^{\frac{1}{2}}} \tag{28}$$

By making use of the relations (cf. (6))

$$\eta + \frac{1}{\eta} = \frac{2}{\sqrt{M^2-1}\tan\omega} \tag{29}$$

$$w_0 = -U\alpha, \qquad t = \tan\omega$$

equation (28) becomes

$$\frac{4}{\alpha}\left(\frac{u}{U}\right) = \frac{4}{E(k')\sqrt{M^2-1}\,(a^2-b^2)^{\frac{1}{2}}} \cdot \frac{(M^2-1)^{\frac{1}{2}}ab(1-k^2)t + (a^2k^2-b^2)}{[(M^2-1)(b^2k^2-a^2)t^2 + 2ab(1-k^2)(M^2-1)^{\frac{1}{2}}t + (a^2k^2-b^2)]^{\frac{1}{2}}}. \tag{30}$$

The slope of the lift curve $dC_L/d\alpha$ is given by the value $4/\alpha(u/U)$ averaged over the wing between the limits $t = \tan \omega_1$, and $t = -\tan \omega_2$. It should be noted that these limits are the roots of the quadratic equation in the denominator of (30). Carrying out the calculation, one finds

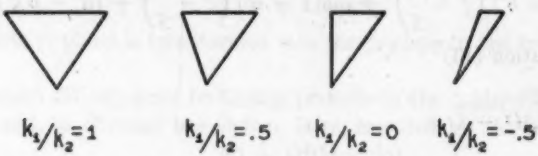

FIG. 5

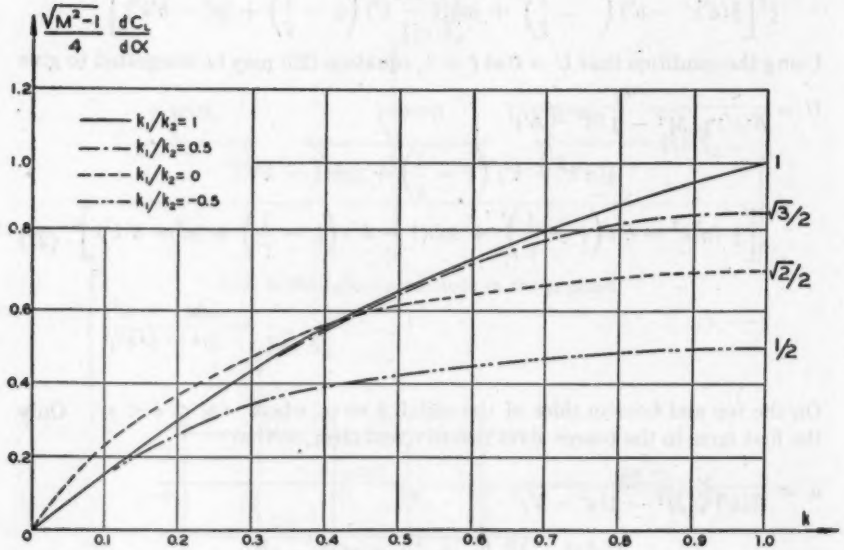

FIG. 6

$$\frac{dC_L}{d\alpha} = \frac{4}{E(k')(M^2-1)(\tan\omega_1+\tan\omega_2)} \frac{a^2-b^2}{(b^2k^2-a^2)^{\frac{1}{2}}} \cdot \left[\sin^{-1}\frac{(M^2-1)^{\frac{1}{2}}(a^2-b^2k^2)t-ab(1-k^2)}{k(a^2-b^2)}\right]_{t=-\tan\omega_2}^{t=\tan\omega_1}. \tag{31}$$

Making use of the identities

$$\begin{aligned}
a^2-b^2 &= 2k_1^2+2k_2^2-2-2k_1^2k_2^2+2(1-k_1k_2)\sqrt{(1-k_1^2)(1-k_2^2)}\\
a+bk &= [2-2k_2^2-2\sqrt{(1-k_1^2)(1-k_2^2)}]/(k_1+k_2)\\
a-bk &= [-2+2k_1^2+2\sqrt{(1-k_1^2)(1-k_2^2)}]/(k_1+k_2)\\
a-bk_1 &= -k_2+k_1^2k_2+k_1\sqrt{(1-k_1^2)(1-k_2^2)}\\
a+bk_2 &= k_1-k_1k_2^2-k_2\sqrt{(1-k_1^2)(1-k_2^2)}\\
k(bk_1-a)-(b-ak_1) &= 0,\ k(bk_2+a)-(ak_2+b)=0
\end{aligned} \tag{32}$$

one may reduce (30) to

$$\frac{\sqrt{M^2-1}}{4}\frac{dC_L}{d\alpha} = \frac{\pi}{2}\frac{k}{E(k')}\left(\frac{k_1+k_2}{k}\right)^{\frac{1}{2}}. \tag{33}$$

It is to be noted that when $k_1 = k_2$

$$\frac{dC_L}{d\alpha} = \frac{2\pi \tan \omega_1}{E(k')} \tag{34}$$

which is the result obtained by Stewart.

In the graph at the end of this paper (Fig. 5) $\frac{1}{4}\sqrt{M^2-1}\, dC_L/d\alpha$ has been plotted against k for several ratios of k_1/k_2. The general shapes of the plan form of the wing for the different ratios are shown in Fig. 6.

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HYPERSONIC FLOW OVER A SLENDER CONE

S. F. Shen

1. Introduction. When the free stream Mach number is very large and the thickness of the body immersed in the stream is very small such that the product of Mach number and the thickness ratio is of the order one, linearization of the differential equations of motion is not possible and the flow characteristics become those of hypersonic flows. The appropriate equations and the similiarity law for both two-dimensional and axially symmetrical flows were derived by H. S. Tsien [1] and their physical significance discussed by W. D. Hayes [2]. The present study is an application of these results to the flow over slender cones, a flow that requires the simplest boundary conditions. The numerical calculation is then not difficult to carry out, and the results can be compared with known related solutions to acquire a proper orientation in this new field of fluid mechanics. The specific quantitative data might be of interest in some practical problems such as the designing of very high velocity projectiles for piercing thick armours.

2. Statement of the Problem and Method of Solution. According to Tsien [1], if M^0 is the free stream Mach number and β is the half angle of the cone (Fig. 1), the hypersonic flow is characterized by a single parameter $K = M^0\beta$. The differential equation is then considerably simpler than the general case. Hayes interpreted this equation by the transformation $x = U \cdot t$ where U is the free stream velocity and has shown that the resultant equation for axially symmetrical flow is that of the propagation of cylindrical waves with t as the time variable. Here the stagnation conditions of wave propagation corresponds to the free stream conditions of the hypersonic flow. The same transformation also gives the appropriate boundary conditions for solving the wave propagation equations. For instance, if $y_0 = f(x)$ represents the value of the radius of the body at x in the hypersonic flow, then $y_0 = f(Ut)$ represents the radius of the expanding or contracting solid surface in the cylindrical wave field. The radial velocity in the hypersonic flow must be such that the flow is tangent to the body at its surface, i.e., $v(y_0) = U\, dy_0/dx$. For the corresponding cylindrical wave field, $v(y_0) = dy_0/dt$. Therefore, in the case of flow over cones, Hayes' interpretation becomes the propagation of cylindrical waves from a uniformly expanding circular cylinder, a problem analogous to the one concerning uniformly expanding spherical waves treated by G. I. Taylor [3]. This is the concept adopted for the present study.

To solve the associated wave problem around a uniformly expanding cylinder, one observes that the radial velocity v, the pressure p, and the density ρ are functions of $s = y/t$ only. That is,

$$\left(\frac{\partial}{\partial t} + \frac{y}{t}\frac{\partial}{\partial y}\right)(v, p, \rho) = 0. \tag{2.1}$$

In the flow over a cone, it means that on every conical surface defined by

$$\theta = \tan^{-1} y/x = \tan^{-1} s/U = s/U, \tag{2.2}$$

v, p and ρ are constant. The existence of such a flow field has been found by both G. I. Taylor [4] and A. Busemann [5].

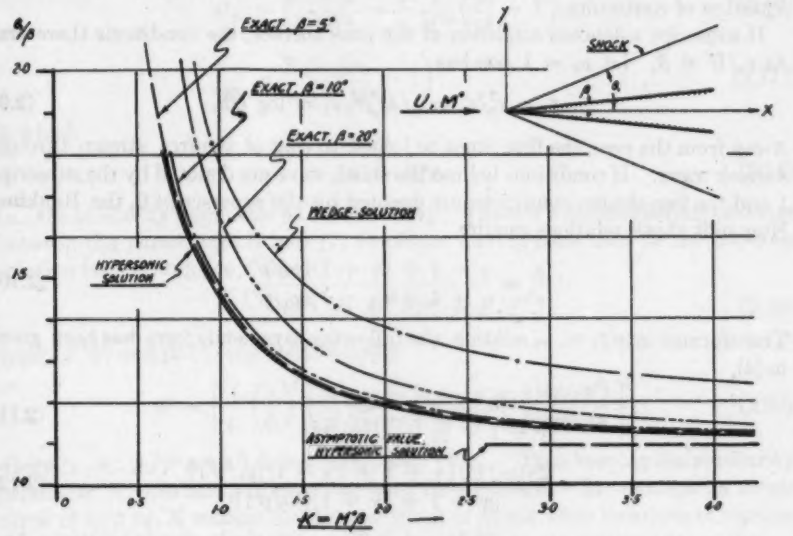

FIG. 1. Shock Wave Angle Vs. K.

Because of eq. (2.1), the equations of equilibrium and continuity become

$$(v - s)\, dv/ds = -\rho^{-1}\, dp/ds. \tag{2.3}$$

$$\left(\frac{v - s}{\rho}\right)\frac{d\rho}{ds} + \frac{dv}{ds} + \frac{v}{s} = 0. \tag{2.4}$$

The flow considered being isentropic,

$$p\rho^{-\gamma} = \text{const.} \tag{2.5}$$

where γ is the ratio of specific heats. Now introduce the following change of variable as was done by Taylor [3]

$$\mu = v/s,\ \zeta = c^2/s^2,\ \sigma = \log s, \tag{2.6}$$

with μ as the new independent variable and c denoting the local velocity of sound. After substitution and simplification, eqs. (2.3) and (2.4) are transformed into

$$\frac{d\zeta}{d\mu} = \frac{2\zeta}{\mu}\frac{\zeta + \{\frac{1}{2}(\gamma + 1)\mu - 1\}(1 - \mu)}{2\zeta - (1 - \mu)^2}, \tag{2.7}$$

$$\frac{d\sigma}{d\mu} = -\frac{1}{\mu}\frac{\zeta - (1 - \mu)^2}{2\zeta - (1 - \mu)^2}. \tag{2.8}$$

These equations are to be solved with the proper boundary conditions in the new variables. A marked resemblance exists between the set of equations (2.7) and (2.8) and the corresponding set in Taylor's solution for wave surrounding an expanding sphere [3]. This is of course to be expected since the only difference of the two is the constant factor in the term involving no derivatives in the equation of continuity.

If subscript s denotes condition at the cone surface, the conditions there are: At $v_s/U = \beta$, i.e. $\mu_s = 1$, one has

$$\zeta_s = c_s^2/v_s^2 = 1/M_s^2\beta^2, \sigma_s = \log U\beta. \tag{2.9}$$

Away from the cone the flow must be joined to that of the free stream through a shock wave. If conditions behind the shock wave are denoted by the subscript 1 and the free stream conditions are denoted by the superscript 0, the Rankine-Hugoniot shock relations require

$$\frac{\rho_1}{\rho^0} = \frac{\gamma - 1 + (\gamma + 1)(p_1/p^0)}{\gamma + 1 + (\gamma - 1)(p_1/p^0)}. \tag{2.10}$$

Transformed into ζ_1 vs. μ_1 relation, the following parametric form has been given in [4],

$$\mu_1 = \frac{2\{(p_1/p^0) - 1\}}{\gamma - 1 + (\gamma + 1)(p_1/p^0)} \tag{2.11}$$

$$\zeta_1 = \frac{2\gamma(p_1/p^0)\{\gamma + 1 + (\gamma - 1)(p_1/p^0)\}}{\{\gamma - 1 + (\gamma + 1)(p_1/p^0)\}^2}. \tag{2.12}$$

Solving the system of eqs. (2.7) and (2.8) is a relatively simple matter. Any of the well-known numerical methods may be applied. The result will be two sets of curves, ζ vs. μ and $(\sigma - \sigma_s) = \log \theta/\beta$ vs. μ, with the quantity $M_s\beta$ as parameter. With any assumed value of $M_s\beta$, the solution can be started with conditions (2.9) till the shock relations (2.11) and (2.12) are satisfied.

Useful physical results of flow problems are always presented as the velocity field, the pressure coefficient at the body and also the location of the shock. These can be computed from the sets of curves obtained above. The shock location is directly given on the $(\sigma - \sigma_s)$ vs. μ curve,

$$\theta_1/\beta = e^{\sigma_1 - \sigma_s}. \tag{2.13}$$

The velocity variation with θ is as follows:

$$\omega/\beta = v/v_s = \mu\theta/\beta \tag{2.14}$$

where ω = inclination of the local velocity vector. The pressure at cone surface:

$$\frac{p_s}{p^0} = \frac{p_s}{p_1}\frac{p_1}{p^0} = \left(\frac{c_1^2}{c_s^2}\right)^{\gamma/1-\gamma}\frac{p_1}{p^0}. \tag{2.15}$$

But

$$c_1^2/c_s^2 = (\zeta_1/\zeta_s)(\theta_1/\beta)^2 \tag{2.16}$$

so

$$\frac{p_s}{p^0} = \frac{p_1}{p^0}\left[\frac{\zeta_s}{\zeta_1}\left(\frac{\beta}{\theta_1}\right)^2\right]^{\gamma/\gamma-1}. \tag{2.15}'$$

Then,

$$C_{p_s} = \frac{p_s - p^0}{\frac{1}{2}\rho^0 U^2} = \frac{2}{\gamma}\frac{1}{M^{02}}\left(\frac{p_s}{p^0} - 1\right), \tag{2.17}$$

$$\frac{C_{p_s}}{\beta^2} = \frac{2}{\gamma}\frac{1}{K^2}\left(\frac{p_s}{p^0} - 1\right), \tag{2.17}'$$

in which

$$K = M^0\beta, \tag{2.18}$$

i.e., the similarity parameter as defined in [1]. There is a definite correspondence between the parameters K and ζ_s, the latter having been used in the previous solution for convenience. Since

$$K^2\zeta_s = M^{02}/M_s^2 = c_s^2/c^{02} \tag{2.19}$$

using (2.10) and (2.16) one obtains easily

$$K = \left[\frac{1}{\zeta_1}\left(\frac{\beta}{\theta_1}\right)^2\left(\frac{p_1}{p^0}\right)\frac{\gamma + 1 + (\gamma - 1)(p_1/p^0)}{\gamma - 1 + (\gamma + 1)(p_1/p^0)}\right]^{\frac{1}{2}}, \tag{2.20}$$

where ζ_1, θ_1, p_1/p^0 are all determined solely by ζ_s. Thus keeping the similiarity parameter K constant will give the same flow pattern. For example, a single curve of θ_1/β vs. K suffices for the prediction of shock wave locations of various slender cones in hypersonic flows.

3. Numerical Results. The method used in the present study in solving the basic equations (2.7) and (2.8) is by expanding the solution into a series near the initial point (ζ_s, μ_s, σ_s) and using a standard numerical integration thereafter. Assume that near $\mu_s = 1.0$ the solution for (2.7) is

$$\zeta = \sum_{n=0}^{\infty} a_n(\mu - 1)^n = \sum_{n=0}^{\infty} a_n\mu'^n. \tag{3.1}$$

with $\mu' = \mu - 1$, then

$$d\zeta/d\mu = d\zeta/d\mu' = \sum_{n=1}^{\infty} a_n n\mu'^{n-1}. \tag{3.2}$$

Substituting into eq. (2.7) and equating coefficients for the powers of μ' on both sides,

$$\begin{aligned} &a_1 = a_0, \qquad a_2 = -\frac{\gamma - 1}{4}, \qquad a_3 = -\frac{\gamma + 1}{12} \\ &a_4 = \frac{\gamma + 1}{24} - \frac{\gamma - 1}{16a_0}, \qquad a_5 = -\frac{\gamma + 1}{40} - \frac{7 - 3\gamma}{80a_0}, \text{ etc.} \end{aligned} \tag{3.3}$$

It is interesting to see that both a_2 and a_3 are independent of the initial value a_0 ($a_0 = \zeta_s$). By taking terms up to $a_5(\mu - 1)^5$ and taking $\gamma = 1.405$, this series

is used to compute values of ζ for $1 \leq \mu \leq 0.6$. With the parameter ζ_s ranging from 0.3 to 25, the last term of the series taken is no greater than 0.002. The solution is then continued, when necessary, by Adams' method of numerical integration. End point of such curve is determined by graphically finding the intersection with the μ_1 vs. ζ_1 curve.

A similar procedure has been adopted to solve next eq. (2.8). The solution near $\mu_s = 1.0$ is again assumed to be a power series of the form:

$$\sigma = \sum_{n=0}^{\infty} b_n(\mu - 1)^n = \sum_{n=0}^{\infty} b_n \mu'^n. \tag{3.4}$$

$$d\sigma/d\mu = d\sigma/d\mu' = \sum_{n=1}^{\infty} b_n n \mu'^{n-1}. \tag{3.5}$$

Putting into eq. (2.8), with the help of eq. (3.1), one obtains

$$b_1 = -\tfrac{1}{2}, \qquad b_2 = -\tfrac{1}{4}, \qquad b_3 = \frac{1 - a_0}{6a_0},$$
$$b_4 = \frac{-2 + a_0}{8a_0}, \qquad b_5 = \frac{3a_0 - a_0^2 + (\gamma + 1)/4}{10a_0^2}, \text{ etc.} \tag{3.6}$$

As in solving eq. (2.7), the series solution is continued for $\mu < 0.6$ by Adams' method until the shock wave condition is reached.

TABLE 1

ζ and $\sigma - \sigma_s$ vs. μ for various ζ_s's.

ζ_s = value of ζ at $\mu = 1.0$.

μ	ζ	$\sigma - \sigma_s$	ζ	$\sigma - \sigma_s$	ζ	$\sigma - \sigma_s$	ζ	$\sigma - \sigma_s$
1.0	.309	0	.391	0	.510	0	.694	0
0.9	.277	.052	.351	.052	.458	.052	.624	.052
0.8	.245	.105	.310	.107	.406	.108	.553	.109
0.7	.213	.153	.270	.159	.354	.163	.483	.168
0.6	.184	.183	.233	.199	.305	.213	.416	.225
0.5							.359	

μ	ζ	$\sigma - \sigma_s$	ζ	$\sigma - \sigma_s$	ζ	$\sigma - \sigma_s$	ζ	$\sigma - \sigma_s$
1.0	1.000	0	1.563	0	2.778	0	25.0	0
0.9	.899	.052	1.405	.053	2.499	.053	22.499	.053
0.8	.798	.110	1.248	.110	2.220	.111	19.998	.111
0.7	.697	.171	1.091	.174	1.942	.176	17.497	.178
0.6	.600	.234	.937	.242	1.666	.248	15.000	.254
0.5	.510	.305	.790	.320	1.398	.333	12.509	.345
0.4			.658	.402	1.142	.429	10.028	.455
0.3					.909	.535	7.564	.595
0.2					.731	.631	5.129	.787
0.1							2.752	1.088

The computed values of ζ and $\sigma - \sigma_s$ vs. μ at various ζ_s's are tabulated in Table 1. With these information for each ζ_s, one then calculates by means of

eqs. (2.13), (2.15)′, (2.20) and (2.17)′ the wanted shock wave angle θ_1, the pressure at the cone surface p_s/p^0, the corresponding value of K and the pressure coefficient at the cone surface C_{p_s}. The results are summarized in Table 2. Fig. 1 and Fig. 2 show respectively the variation of θ_1/β and C_{p_s}/β^2 as only a func-

TABLE 2

Summary of results

μ_1	ζ_1	$\sigma_1 - \sigma_s$	θ_1/β	p_s/p^0	K	C_{p_s}/β^2*
.244	.794	.590	1.804	1.910	.66	2.95
.394	.654	.408	1.504	2.549	.92	2.65
.522	.529	.290	1.337	3.568	1.22	2.45
.618	.430	.217	1.242	5.093	1.59	2.31
.691	.352	.169	1.184	7.584	2.10	2.20
.747	.291	.134	1.143	12.320	2.74	2.14
.790	.244	.111	1.117	24.434	4.00	2.10

* These are smoothed out values.

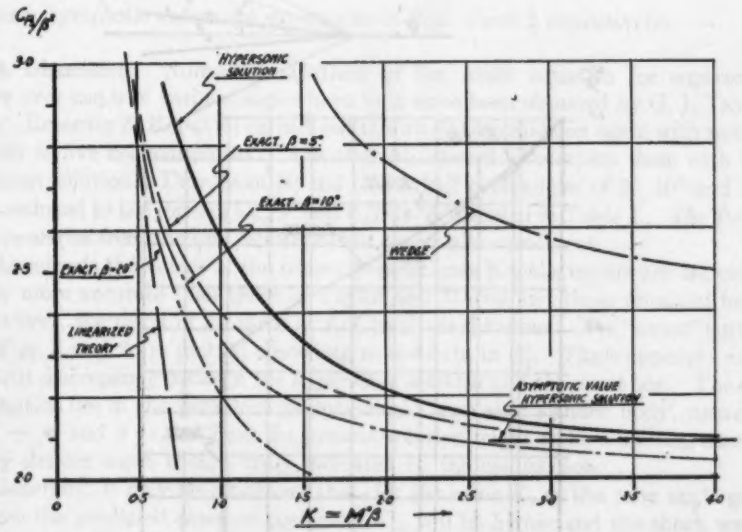

FIG. 2. Pressure At Cone Surface Vs. K.

tion of the parameter K. The θ_1/β curve is fairly smooth, but for C_{p_s}/β^2 there appears some scattering of the points, which could be attributed to the cumulative errors in the numerical and graphical solutions. The velocity distribution as represented by eq. (2.14) has been evaluated for three values of K. The variations of ω/β vs. θ/β are given in Table 3 and Fig. 3.

As K goes to very large values, the asymptotic values of θ_1/β and C_{p_s}/β^2 can

TABLE 3

Local flow direction within shock

$K = 0.66$	
θ/β	ω/β
1.000	1.000
1.054	.949
1.117	.894
1.192	.834
1.282	.769
1.395	.698
1.536	.614
1.708	.512
1.880	.376

$K = 1.22$	
θ/β	ω/β
1.000	1.000
1.054	.949
1.083	.921
1.116	.893
1.187	.831
1.266	.760
1.358	.679

$K = 4.00$	
θ/β	ω/β
1.000	1.000
1.027	.975
1.053	.948
1.082	.920
1.111	.891
1.140	.855

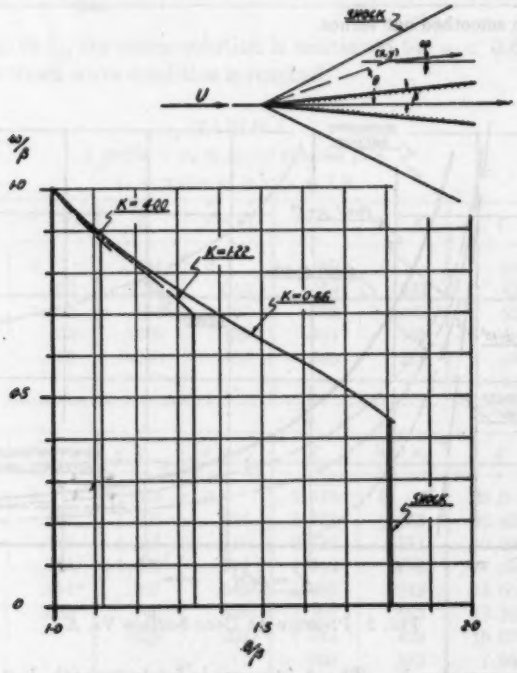

FIG. 3. Local Flow Direction

be derived as follows. Large K means large expansion velocity of the cylindrical wave problem, which in turn means a strong shock. A strong shock leads to large value of p_1/p^0. If one lets $p_1/p^0 \to \infty$, the shock wave conditions (2.11)

and (2.12) reduce to

$$\mu_1 = \frac{2}{\gamma + 1} = 0.8316, \qquad \zeta_1 = \frac{2\gamma(\gamma - 1)}{(\gamma + 1)^2} = 0.1968. \tag{3.7}$$

Because μ_1 is very close to 1.0, it is possible to calculate directly the value of the parameter ζ_s for $K \to \infty$. Taking terms up to $(\mu - 1)^3$ in the series (3.1), one gets

$$a_0 = \zeta_s = 0.239 \tag{3.8}$$

ζ_s having been determined, taking terms up to $(\mu - 1)^3$ in the series (3.4) yields

$$\sigma_1 - \sigma_s = 0.089$$

or

$$(\theta_1/\beta)_{k\to\infty} = e^{0.089} = 1.093. \tag{3.9}$$

To calculate C_{p_s}, it can be derived from (2.17)′, (2.15)′ and (2.20) that, as $p_1/p^0 \to \infty$,

$$\left(\frac{C_{p_s}}{\beta^2}\right)_{k\to\infty} = \frac{2}{\gamma}\frac{\gamma + 1}{\gamma - 1}\left(\frac{p_s}{p_1}\right)^{1/\gamma}\zeta_s = 2.09. \tag{3.10}$$

These asymptotic values are shown also in Figs. 1 and 2 respectively.

4. Discussion. Numerical solutions of the exact equation for supersonic flow over cones of various angles have long since been obtained by G. I. Taylor [4]. Recently Z. Kopal [6] carried out the whole computation again with results given to five decimal places. It is of great interest to compare them with the present solution. Data from [4] and [6] for half cone angles of 5°, 10° and 20° are reduced to the form of C_{p_s}/β^2 and θ_1/β vs. K as listed in Table 4. The shock wave angles from the two references are practically coincident.

As regards the values of the pressure coefficients Kopal's results are undoubtedly more accurate than those of Taylor and Macoll and those obtained here. However, the trend of all three sets of results is the same. The "exact" curves in Figs. 1 and 2 are plotted according to the data in [6]. There appears a consistent discrepancy between the hypersonic solution and the exact one. The explanation lies in the definition of hypersonic flow over slender body, namely, $M^0 \to \infty$ and $\beta \to 0$. That the present solution ought to be a limiting case of very slender cones is then truly indicated by the comparison.

Generally, it may be concluded that, for the same K, as the cone angle gets larger the predicted pressure coefficient C_{p_s} will be higher and the shock wave angle θ_1 smaller than the exact solution. When the half cone angle is 10°, at $K = 1.0$ the errors in C_{p_s} and θ_1 are roughly 10% and 2% respectively. With increasing Mach number the errors go down.

A comparison with the linearized supersonic flow solution bears out the marked difference in behavior of a hypersonic flow. Obtained by Th. v. Kármán [7], the linearized solution for slender cones gives

$$\frac{C_{p_s}}{\beta^2} = \log\frac{2}{\sqrt{M^{0^2} - 1}\,\beta}, \tag{4.1}$$

TABLE 4

Cp_s/β^2 and θ_1/β vs. K from exact solutions

β = .0873

Kopal's values (ref. 6)		
K	C_{p_s}/β^2	θ_1/β
.517	2.87	2.18
.647	2.66	1.84
.741	2.56	1.69
.910	2.47	1.54
1.151	2.33	1.38
1.965	2.18	1.20
∞	2.085	1.094

β = .1763†

Kopal's values (ref. 6)			Taylor's values (ref. 4)		
K	C_{p_s}/β^2	θ_1/β†	K	C_{p_s}/β^2	θ_1/β†
.421	3.08	2.81	.319	3.48	3.88
.490	2.99	2.41	.421	3.09	2.80
.583	2.71	2.02	.587	2.90	2.08
.720	2.53	1.79	.963	2.45	1.56
.956	2.37	1.50			
1.605	2.16	1.26			
2.67	2.08	1.16			
∞	2.02	1.10			

β = .364†

Kopal's values (ref. 6)			Taylor's values (ref. 4)		
K	C_{p_s}/β^2	θ_1/β†	K	C_{p_s}/β^2	θ_1/β†
.681	2.54	2.29	.601	2.69	2.69
.776	2.39	2.00	.775	2.40	2.00
.890	2.28	1.82	1.041	2.23	1.61
1.034	2.18	1.61	1.518	2.02	1.35
1.228	2.09	1.47	3.545	1.93	1.16
1.512	2.02	1.35			
2.02	1.95	1.25			
3.49	1.89	1.16			
∞	1.85	1.11			

† The tangents of θ_1 and β are taken when angles are no longer very small.

which in hypersonic region, if applicable, reduces to

$$C_{p_s}/\beta^2 = \log 2/K. \qquad (4.1)'$$

One obvious absurdity is that when $K > 2$ the pressure at the cone surface becomes negative. The curve for smaller values of K has been plotted in Fig. 2. The error grows extremely fast when K goes beyond 0.5. But it is also of

interest to see that in the neighborhood of $K = 0.5$, the linearized theory actually gives better agreement with the exact solution than the hypersonic one for moderate cone angles.

According to v. Kármán [7], the limiting case of hypersonic flow with $\gamma = 1$ and $K \to \infty$ is particularly simple in that the shock wave angle is equal to the cone angle. The same conclusion can be obtained by referring to eq. (3.7) and the series (3.1), which give $\mu_1 = 1$ and $\zeta_1 = \zeta_0$. The eq. (2.16) then shows that $\theta_1/\beta = 1$, since c_1^2/c_∞^2 is unity when $\gamma = 1$. Physically, the density of fluid over the surface of the cone is now infinite and the fluid is not influenced by the presence of the body till it "hits" the surface. After it strikes the surface it flows along the surface. Consider any element of surface indicated in Fig. 4.

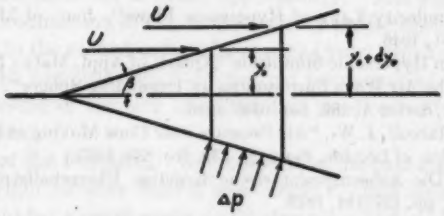

FIG. 4.

The mass rate of flow into this element is $2\pi y_0\, dy_0 \rho^0 U$. After striking the surface, the normal velocity component $U\beta$ is lost. Therefore if Δp is the pressure increment at the surface,

$$2\pi y_0\, dy_0 \cdot \rho^0 U \cdot U\beta = 2\pi y_0\, dy_0 \beta^{-1} \cdot \Delta p, \tag{4.2}$$

or,

$$(C_{p_0}/\beta^2)_{k\to\infty} = 2. \tag{4.2}'$$

Thus in the case of cone, the difference for $\gamma = 1$ and $\gamma = 1.405$ of the limiting values of C_{p_0} as $K \to \infty$ is only 5%.

Another comparison can be made with the hypersonic flow over wedges. The latter results are obtainable by simply considering an oblique shock wave when the Mach number is large and the wedge angle small, therefore the shock wave angle itself is small. In previous notations, the shock wave angle and the pressure coefficient at the wedge are given by

$$\frac{\theta_1}{\beta} = \frac{\gamma+1}{4} + \sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{K^2}} \tag{4.3}$$

and

$$C_{p_0}/\beta^2 = 2\theta_1/\beta. \tag{4.4}$$

Both θ_1/β and C_{p_0}/β^2 are much higher than those for the cones, as shown in Figs. 1 and 2. This indicates the difference in behavior of two dimensional and three dimensional bodies.

With regard to the practical application of the present solution, one must be aware of the fact that the boundary layer effect has been entirely neglected. Although Maccoll claimed excellent agreement of measured data from projectile experiment with Taylor's solution [8], it is highly improbable that as the Mach number further increases the same could be true for very slender cones. The thickness of the shock wave layer, being of the order of 10% of the cone thickness, becomes then so thin that the presence of a viscous boundary layer would be felt. The effect on cones of moderate apex angle probably would not be too great. However, investigations have to be made before any conclusion on this point could be drawn.

REFERENCES

1. Tsien, H. S., "Similarity Laws of Hypersonic Flows", Jour. of Math. and Phys., **25**, pp. 247–251, 1946.
2. Hayes, W. D., "On Hypersonic Similitude", Quart. of Appl. Math., **5**, p. 105, 1947.
3. Taylor, G. I., "The Air Wave Surrounding an Expanding Sphere", Proc. of Royal Soc. of London, Series A, **186**, No. 1006, 1946.
4. Taylor, G. I., & Maccoll, J. W., "Air Pressure over Cone Moving at High Speeds", Proc. of Royal Soc. of London, Series A, **139**, No. 838, 1933.
5. Busemann, A., "Die Achsensymmetrische Kegelige Überschallströmung", Luftfahrtforschung, pp. 137–144, 1942.
6. Kopal, Z., "Tables of Supersonic Flow around Cones", Tech. Rept. No. 1, M.I.T. Dept. of Elect. Eng'g., Center of Analysis, 1947.
7. V. Kármán, Th., "The Problem of Resistance in Compressible Fluids", Proc. of the 5th Volta Congress, pp. 275–277, Rome, 1936.
8. Maccoll, J. W., "The Conical Shock Wave Formed by a Cone Moving at a High Speed", Proc. of Royal Soc. of London, Series A, **159**, No. 898, 1937.

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ON SUPERSONIC FLOW PAST A SLIGHTLY YAWING CONE*

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Introduction. This paper is concerned with the motion of a circular cone, of not too blunt an angle, through air at high speed. If the direction of motion of the cone coincides with its axis of symmetry, the resulting air flow is well known.[1] Here we consider the perturbation produced by a small "yaw"—i.e., the case in which the cone is moving not quite in the direction of its axis. The results are confirmed experimentally, and have applications to ballistics, though we are not concerned with the latter here; they may also be useful as providing a check on various approximate methods of wider applicability. The square of the yaw is neglected—an approximation of which the validity is discussed. (Similar methods can be applied to the second-order effects of the yaw, which are also of ballistic significance; but the computations have not yet been completed.) It should be observed that, because of the lack of symmetry, the flow will be neither irrotational nor isentropic.

1. Formulation of the Problem. We consider, in the first instance, a projectile having as head a right circular cone of semi-angle θ_s, and moving with velocity U in a direction making a small angle ϵ with its axis. If $\theta_s < 56°$ and the Mach number U/a (where a denotes the velocity of sound) is large enough, it is known[2] that the projectile will be preceded by a conical shock-wave, attached to the solid cone at their common vertex. (The shock-wave cone is not assumed to be circular, though it turns out to be so.) An expansion wave originates at the shoulder of the projectile (see Fig. 1); and we shall be concerned with the flow in the region between the shock-wave and the expansion wave, with the object of determining the aerodynamic forces on the head of the projectile, and the configuration of the shock-wave. It will be assumed that the Mach number is large enough for the flow in this region to be locally supersonic;[3] it is then independent of the conditions outside the region, so that we may for convenience suppose that the conical portion of the projectile extends to infinity.

We consider the motion relative to the (solid) cone, and use spherical coordinates (r, θ, ϕ) in which the origin $(r = 0)$ is the cone vertex, the axis $\theta = 0$ is in the direction of the velocity U of the free air-stream, and the plane $\phi = 0$ is the plane through this axis and the axis of symmetry of the cone (the "*plane of yaw*").

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[1] G. I. Taylor and J. W. Maccoll, Proc. Roy. Soc. A 139 (1933) pp. 278–311; J. W. Maccoll, Proc. Roy. Soc. A 159 (1937) pp. 459–472; A. Busemann, Luftfahrtforschung 19 (1942) pp. 137–144.

[2] See Taylor and Maccoll, loc. cit. in footnote (1).

[3] The lower bound for U/a depends on θ_s; for $\theta_s = 15°$ it is 1.2. See Taylor and Maccoll, loc. cit., for further values.

We use u, v, w to denote the components, in the directions of increasing r, θ, ϕ respectively, of the air velocity at the point (r, θ, ϕ) between the shock-wave and the cone; and the pressure and density are denoted by p and ρ. The pressure and density of the free air-stream are denoted by p_1 and ρ_1; thus $a^2 = \gamma p_1/\rho_1$.

As usual, it is assumed that U is constant in magnitude and direction, that the direction of the axis of symmetry of the projectile is constant, and that the effects of viscosity, heat transfer, the boundary layer and skin friction are negligible. These idealizations are known to give good results if the yaw is zero, and may be expected to apply to the general case likewise. Besides assuming the "yaw" ϵ to be small, we make the assumption (plausible, but difficult to justify mathematically with complete rigor) that the flow depends *continuously* on ϵ.

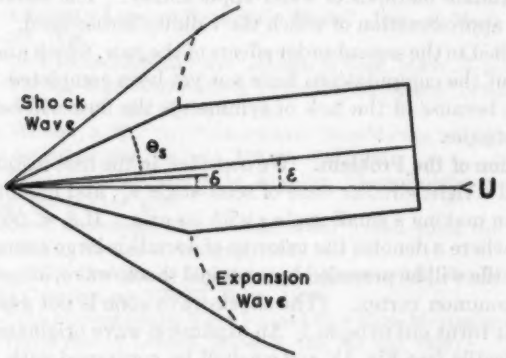

FIG. 1

2. Non-Yaw Case. Let $\bar{u}, \bar{v}, \bar{w}, \bar{p}, \bar{\rho}$, denote the velocity-components, pressure and density for the case in which the yaw ϵ is zero; and let the shock-wave surface have equation $\theta = \theta_s$ then. For convenience we summarize the equations relating these quantities.[2] It is found that $\bar{u}, \bar{v}, \cdots, \bar{\rho}$ are (for a given θ_s and Mach number) functions of θ alone, and $\bar{w} = 0$. The equations of motion and of continuity give

$$\bar{v} = \bar{u}', \quad \bar{p}'/\bar{p} = -\bar{u}'(\bar{u} + \bar{u}'') \tag{1}$$

$$\bar{u}'' + \{\cot\theta + (ln\bar{\rho})'\}\bar{u}' + 2\bar{u} = 0 \tag{2}$$

where a dash ($'$) is used to denote differentiation with respect to θ. The flow is adiabatic, and hence "isentropic", i.e.,

$$\bar{p}/\bar{\rho}^\gamma \text{ is constant} \tag{3}$$

where γ is the ratio of specific heats ($= 1.405$). The Bernoulli equation gives

$$\tfrac{1}{2}(\bar{u}^2 + \bar{v}^2) + \gamma\bar{p}/(\gamma - 1)\bar{\rho} = \text{constant} = \tfrac{1}{2}c^2 \tag{4}$$

where $c^2 = U^2 + 2a^2/(\gamma - 1)$.

From these equations one obtains

$$(ln\ \bar{\rho})' = -\frac{2\bar{u}'(\bar{u} + \bar{u}'')}{(\gamma - 1)(c^2 - \bar{u}^2 - (\bar{u}')^2)} \tag{5}$$

The boundary conditions are:

When $\theta = \theta_s$,

$$\bar{u}' = 0 \tag{6}$$

When $\theta = \theta_w$,

$$\begin{aligned} &\bar{u} = U\cos\theta \\ &U\rho_I \sin\theta + \bar{\rho}\bar{v} = 0 \\ &\bar{p} - p_I = U\rho_I \sin\theta(\bar{v} + U\sin\theta) \\ &U^2\sin^2\theta = \{(\gamma - 1)p_I + (\gamma + 1)\}/2\rho_I \end{aligned} \tag{7}$$

The additional relations:

When $\theta = \theta_w$,

$$\bar{u}' = -\{(\gamma - 1)U\sin\theta + 2a^2/U\sin\theta\}/(\gamma + 1)$$

and

$$ln\bar{\rho}' = -2\cot\theta/(\gamma + 1) \tag{8}$$

follow easily, and will be useful. (We shall, for short, write $\bar{\rho}'/\bar{\rho} = (ln\bar{\rho})'$ as $ln\bar{\rho}'$.)

These equations determine the non-yaw quantities $\bar{u}, \cdots, \bar{\rho}$, and θ_w (for given θ_s and Mach number), and we shall regard them as known.[4] It should however be observed that there are *two* régimes (with different values of θ_w) which satisfy these equations. Only one of these is observed in ballistic practice, and the symbols $\bar{u}, \cdots, \theta_w$ will refer to this one. Why the other régime does not so occur is not known (it has been observed in wind-tunnel experiments); but we shall see that the present theory throws some light on this question.

3. Differential Equations. In the general case, the equation of the shock-wave surface will involve ϕ as well as θ; but its observed rectilinearity shows that it is independent of r. It follows that the whole flow is independent of r; thus we may expand

$$\left.\begin{aligned} u &= \bar{u} + \sum(x_n\cos n\phi + X_n\sin n\phi) \quad (X_0 = 0) \\ v &= \bar{v} + \sum(y_n\cos n\phi + Y_n\sin n\phi) \quad (Y_0 = 0) \\ w &= \quad \sum(z_n\sin n\phi - Z_n\cos n\phi) \quad (Z_0 = 0) \\ p &= \bar{p} + \sum(\eta_n\cos n\phi + H_n\sin n\phi) \quad (H_0 = 0) \\ \rho &= \bar{\rho} + \sum(\xi_n\cos n\phi + \Xi_n\sin n\phi) \quad (\Xi_0 = 0) \end{aligned}\right\} \tag{9}$$

[4] Extensive tabulations are given in vol. 1, Technical Reports, Center of Analysis, Massachusetts Institute of Technology.

where the summation is over $n = 0, 1, 2, \cdots$, and the quantities x_n, X_n etc. are small and are (for fixed θ_s, ϵ and Mach number) functions of θ only. Similarly the equation of the shock-wave will be

$$\theta = \theta_w + \sum(\alpha_n \cos n\phi + A_n \sin n\phi) \quad (A_0 = 0) \tag{10}$$

where α_n, A_n are small constants. We shall see that only a finite number of these quantities are non-zero, so that questions of convergence and term-by-term differentiability do not arise.[5] It will be assumed that ϵ is small enough for the products of the quantities x_n, etc., to be negligible; as will be shown, this amounts to assuming that ϵ^2 is negligible (in practice, say $\epsilon \leqq 5°$).

The equations of motion $\left(\dfrac{d}{dt}(u, v, w) = -\dfrac{1}{\rho}\nabla p\right)$ give, for spherical coordinates and a steady flow independent of r,

$$\left.\begin{aligned} v\frac{\partial u}{\partial \theta} + w \operatorname{cosec}\theta \frac{\partial u}{\partial \phi} - v^2 - w^2 &= 0 \\ v\frac{\partial v}{\partial \theta} + w \operatorname{cosec}\theta \frac{\partial v}{\partial \phi} + \frac{1}{\rho}\frac{\partial p}{\partial \theta} + uv - w^2 \cot\theta &= 0 \\ v\frac{\partial w}{\partial \theta} + w \operatorname{cosec}\theta \frac{\partial w}{\partial \phi} + \frac{\operatorname{cosec}\theta}{\rho}\frac{\partial p}{\partial \phi} + uw + vw \cot\theta &= 0 \end{aligned}\right\} \tag{11}$$

Similarly, the equation of continuity $(\nabla\{\rho(u, v, w)\} = 0)$ gives

$$2\rho u \sin\theta + \frac{\partial}{\partial \theta}(\rho v \sin\theta) + \frac{\partial}{\partial \phi}(\rho w) = 0. \tag{12}$$

The "equation of state" needs more consideration. The entropy will now be altered at the shock-wave by amounts which vary with the angle of incidence, and hence with ϕ. But since the flow is adiabatic and steady, we have along each streamline

$$\frac{\gamma p}{\rho} = \frac{dp}{d\rho} = \frac{\dfrac{\partial p}{\partial \theta}d\theta + \dfrac{\partial p}{\partial \phi}d\phi}{\dfrac{\partial \rho}{\partial \theta}d\theta + \dfrac{\partial \rho}{\partial \phi}d\phi} = \frac{v\dfrac{\partial p}{\partial \theta} + w \operatorname{cosec}\theta \dfrac{\partial p}{\partial \phi}}{v\dfrac{\partial \rho}{\partial \theta} + w \operatorname{cosec}\theta \dfrac{\partial \rho}{\partial \phi}};$$

and, neglecting the products of the small terms w, $\partial p/\partial\phi$ and $\partial\rho/\partial\phi$ we obtain

$$p^{-1}\partial p/\partial\theta = (\gamma/\rho)(\partial\rho/\partial\theta)$$

so that $ln\ p - \gamma ln\rho$ is independent of θ as well as of r. Thus, on using (3), we may write

$$ln(p/\bar{p}) - \gamma ln(\rho/\bar{\rho}) = \sum(d_n \cos n\phi + D_n \sin n\phi) \tag{13}$$

[5] In fact, all the quantities x_n, X_n, etc., vanish except for x_1, y_1, z_1, ξ_1, η_1 and α_1. The extra terms are retained because they give no more trouble, and enable the validity of the present approximation to be investigated; cf. §7.

where d_n , D_n are constants (i.e., are independent of r, θ and ϕ), and $D_0 = 0$, for convenience.

Although it is not independent of the above equations, it is convenient to use also the Bernoulli equation. Even though the flow is not irrotational, we have that along each streamline the expression $\frac{1}{2}(u^2 + v^2 + w^2) + \gamma p/(\gamma - 1)\rho$ is constant; and since this streamline constant is unchanged on crossing a shock-wave[6], we obtain the energy equation

$$\tfrac{1}{2}(u^2 + v^2 + w^2) + \gamma p/(\gamma - 1)\rho = \tfrac{1}{2}U^2 + \gamma p_I/(\gamma - 1)\rho_I . \tag{14}$$

We now substitute from (9) in equations (11) – (14). To illustrate the process, we deal with the first of equations (11) in detail; the others are treated similarly, though less simply. Since the terms $w\partial u/\partial\phi$ and w^2 are negligible, the first of equations (11) reduces to $v(\partial u/\partial\theta - v) = 0$, and so, from (9),

$$\bar{u}' + \sum(x_n' \cos n\phi + X_n' \sin n\phi) = \bar{v} + \sum(y_n \cos n\phi + Y_n \sin n\phi).$$

The terms which do not vanish with ϵ must, of course, cancel (compare equation (1)); and we are left with an identity in ϕ in which the corresponding coefficients of $\cos n\phi$ and, if $n \neq 0$, of $\sin n\phi$, may be equated. Thus, since $X_0 = 0 = Y_0$, we obtain for every n $(= 0, 1, 2, \cdots)$

$$x_n' = y_n , \quad X_n' = Y_n \tag{15}$$

In a similar way, the other equations (11) – (14) give

$$\bar{u}'y_n' + (\bar{u}'' + \bar{u})y_n + \bar{u}'x_n + \eta_n'/\bar{\rho} - \xi_n\bar{p}'/\bar{p}^2 = 0 \tag{16}$$

$$\bar{u}'z_n' + (\bar{u} + \bar{u}' \cot\theta)z_n - n\eta_n/\bar{\rho} \sin\theta = 0 \tag{17}$$

$$y_n' + (\cot\theta + ln\bar{\rho}')y_n + 2x_n + nz_n \operatorname{cosec}\theta + \bar{v}(\xi_n/\bar{\rho})' = 0 \tag{18}$$

$$\eta_n/\bar{p} - \gamma\xi_n/\bar{p} = d_n \tag{19}$$

$$\bar{u}'x_n' + \bar{u}x_n + \eta_n/\bar{\rho} + d_n\bar{p}/(\gamma - 1)\bar{p} = 0 \tag{20}$$

together with just the same equations connecting the capital letters (i.e., with x_n , y_n , $\cdots$, d_n replaced throughout by X_n , Y_n , $\cdots$, D_n respectively).

Before reducing these equation to a form suitable for computation, it is convenient to consider the boundary conditions.

4. Boundary Conditions at the Shock-Wave. Let n be the unit-vector normal to the shock-wave surface at a general point of it, and let t_1 , t_2 , be perpendicular tangent vectors there. We use ν to denote the normal velocity ($= (u,v,w)\cdot n$), and suffixes I, II to refer to the two sides of the shock-wave (side I being in the undisturbed uniform air stream, as before). The Rankine-Hugoniot conditions give[7] that, at the shock-wave,

[6] This is a consequence of the Rankine-Hugoniot shock-wave equations; cf. §4.

[7] See, for example, "Aerodynamic Theory", ed. W. F. Durand, vol. 3, pp. 237–238 (Berlin 1935).

$$(U\cos\theta, -U\sin\theta, 0)\cdot t_1 = (u,v,w)\cdot t_1 \tag{21}$$

$$(U\cos\theta, -U\sin\theta, 0)\cdot t_2 = (u,v,w)\cdot t_2 \tag{22}$$

$$\rho_I v_I = \rho_{II} v_{II} \tag{23}$$

$$p_{II} - p_I + \rho_I v_I (v_{II} - v_I) = 0 \tag{24}$$

$$v_{II} = \{(\gamma - 1)v_I + 2a^2/v_I\}/(\gamma + 1) \tag{25}$$

(respectively expressing the continuity of the tangential velocity and the equations of continuity, impulse and energy).

From (10), a normal vector to the shock-wave surface is

$$\nabla\{\theta - \theta_w - \sum(\alpha_n \cos n\phi + A_n \sin n\phi)\};$$

hence we may take

$$n = \{0, 1, \operatorname{cosec}\theta \sum n(\alpha_n \sin n\phi - A_n \cos n\phi)\} \tag{26}$$

(which is a unit-vector to the present approximation), and hence may take

$$t_1 = (1,0,0) \tag{27}$$

$$t_2 = t_1 \times n = \{0, -\operatorname{cosec}\theta \sum n(\alpha_n \sin n\phi - A_n \cos n\phi), 1\}. \tag{28}$$

On substituting from (9) in (21) we obtain that, when

$$\theta = \theta_w + \sum(\alpha_n \cos n\phi + A_n \sin n\phi), \bar{u} + \sum(x_n \cos n\phi + X_n \sin n\phi) = U\cos\theta,$$

and thus that, when $\theta = \theta_w$ (on expanding by Taylor's theorem and neglecting products of small quantities),

$$\bar{u} + \bar{u}'\sum(\alpha_n \cos n\phi + A_n \sin n\phi) + \sum(x_n \cos n\phi + X_n \sin n\phi)$$
$$= U\{\cos\theta - \sin\theta \sum(\alpha_n \cos n\phi + A_n \sin n\phi)\}.$$

The terms which do not vanish with ϵ must cancel (cf. (7)); and, on equating the coefficients of $\cos n\phi$ and $\sin n\phi$ (and remembering that $X_0 = 0 = A_0$), we have:

When $\theta = \theta_w$,

$$x_n = -\alpha_n(\bar{u}' + U\sin\theta), \qquad X_n = -A_n(\bar{u}' + U\sin\theta) \tag{29}$$

In a similar way it can be shown, from equations (22) – (25), that when $\theta = \theta_w$ we have

$$z_n \sin\theta = n\alpha_n(\bar{v} + U\sin\theta),$$

$$\alpha_n(-\bar{v}\cot\theta + \bar{v}' + \bar{v}\, ln\bar{\rho}') + y_n + \xi_n \bar{v}/\bar{p} = 0,$$

$$\alpha_n(\bar{p}'/\bar{p}\bar{v} + \bar{v}' + \bar{v}\cot\theta + 2U\cos\theta) + y_n + \eta_n/\bar{p}\bar{v} = 0,$$

$$\alpha_n\{\bar{v}' - 2a^2\cot\theta/(\gamma + 1)U\sin\theta + (\gamma - 1)U\cos\theta/(\gamma + 1)\} + y_n = 0,$$

together with the same equations with capital letters (A_n, X_n, etc.) replacing the small letters (α_n, x_n, etc.) throughout. These equations may now be simpli-

fied by using properties of the non-yaw solution ((1), (2), (7)); further, (29) may be used to solve them in terms of α_n, and we obtain:

When $\theta = \theta_w$,

$$nx_n + z_n \sin\theta = 0 \tag{30}$$

$$y_n = 2\alpha_n(2\bar{u} - \bar{u}' \cot\theta)/(\gamma + 1) \tag{31}$$

$$\xi_n/\bar{p} = 2\alpha_n\{(\gamma - 1)\bar{u} + (\gamma + 2)\bar{u}' \cot\theta\}/(\gamma + 1)\bar{u}' \tag{32}$$

$$\eta_n/\bar{p}\bar{u}' = -\alpha_n\{(\gamma + 5)\bar{u} + (\gamma - 1)\bar{u}' \cot\theta\}/(\gamma + 1) \tag{33}$$

together with the same equations connecting the capital letters.

5. Boundary Conditions at the Solid Cone. At the surface of the solid cone, the normal velocity $\nu = (u,v,w)\cdot\mathbf{n} = 0$. Now, the equation to the cone surface is readily found to be

$$\cos\theta\cos\epsilon + \sin\theta\sin\epsilon\cos\phi = \cos\theta_s,$$

which on neglecting ϵ^2 reduces to

$$\theta = \theta_s + \epsilon\cos\phi \tag{34}$$

to which equation (10) would reduce if α_1 were ϵ and all other α's and A's were 0. Thus, making these replacements in (26), we obtain $\mathbf{n} = (0, 1, \epsilon\sin\phi\,\mathrm{cosec}\,\theta)$; and so the condition is that, when $\theta = \theta_s + \epsilon\cos\phi$, $v + \epsilon w\sin\phi\,\mathrm{cosec}\,\theta = 0$—i.e., $v = 0$, since ϵw is negligible. Hence, when $\theta = \theta_s$ we have $v + v'\epsilon\cos\phi = 0$, which gives:

When $\theta = \theta_s$, $Y_m = 0 = y_n$ if $n \neq 1$;

$$y_1 = -\epsilon\bar{v}' = 2\epsilon\bar{u} \tag{35}$$

(on using (1), (2) and (6)).

6. Reduction of the Equations. Since the equations for the capital letters are the same as those for the small letters (except for (35)), we can restrict our attention to the latter. We define

$$t_n = nx_n + z_n\sin\theta \tag{36}$$

Then the result of eliminating η_n between (17) and (20) can be written

$$\bar{u}'t_n' + \bar{u}t_n + nd_n\bar{p}/(\gamma - 1)\bar{p} \tag{37}$$

The usual integrating factor is (on using (2))

$$\frac{1}{\bar{u}'}\exp\left\{\int\frac{\bar{u}}{\bar{u}'}\,d\theta\right\} = (-\bar{u}'\bar{p}\sin\theta)^{-1}/\bar{u}'$$

(It should be observed that $\bar{u}'$ is negative.) Thus (37) can be written

$$\{t_n(-\bar{u}'\bar{p}\sin\theta)^{-1}\}' + nd_n\bar{p}(-\bar{u}'\bar{p}\sin\theta)^{-1}/(\gamma - 1)\bar{p}\bar{u}' = 0$$

whence, since $t_n = 0$ when $\theta = \theta_w$ (from (30)), we obtain

$$nx_n + z_n \sin\theta = t_n = -\{nd_n/(\gamma-1)\}T(\theta)$$

$$\text{where } T(\theta) = (-\bar{u}'\bar{p}\sin\theta)^{\frac{1}{2}} \int_{\theta_w}^{\theta} \bar{p}d\theta/\bar{\rho}\bar{u}'(-\bar{u}'\bar{p}\sin\theta)^{\frac{1}{2}} \tag{38}$$

When θ approaches θ_s the integral here becomes infinite, since $\bar{u}'$ vanishes; nevertheless $T(\theta)$ has a finite limit. This might not be easy to compute numerically from (38) directly; but it is easy to see from (38) that as $\theta \to \theta_s$ we have $T'(\theta) = O(\theta - \theta_s)^{-\frac{1}{2}}$, and so (37) gives:

$$\text{When } \theta = \theta_s,\ t_n = -nd_nT(\theta_s)/(\gamma-1), \text{ where } T(\theta_s) = \bar{p}/\bar{\rho}\bar{u} \tag{39}$$

In equation (18) we can now express all the unknown variables in terms of x_n and its derivatives. In fact, $y_n = x_n'$ (15), z_n is given by (38), and from (19) and (20) we obtain

$$\xi_n/\bar{p} = -\bar{p}(\bar{u}'x_n' + \bar{u}x_n)/\gamma\bar{p} - d_n/(\gamma-1)$$

and so (after simplifying by using (3) and (1))

$$(\xi_n/\bar{p})' = ln\bar{p}'[\{\bar{u}'x_n'' + (\bar{u}'' + \bar{u})x_n' + \bar{u}'x_n\} - (\gamma-1)(\bar{u}'x_n' + \bar{u}x_n)ln\bar{p}']/\bar{u}'(\bar{u}+\bar{u}'').$$

Thus (18) gives

$$\begin{aligned} x_n''\{\bar{u}'' + \bar{u}'ln\bar{p}' + \bar{u}\} + x_n'\{(\bar{u}+\bar{u}'')(\cot\theta + 2ln\bar{p}') - (\gamma-1)\bar{u}'(ln\bar{p}')^2\} \\ + x_n\{(2 - n^2\operatorname{cosec}^2\theta)(\bar{u}+\bar{u}'') + \bar{u}'ln\bar{p}' - (\gamma-1)\bar{u}(ln\bar{p}')^2\} \\ - n^2\operatorname{cosec}^2\theta\, d_n(\bar{u}+\bar{u}'')T(\theta)/(\gamma-1) = 0 \end{aligned}$$

To simplify this we first eliminate $\bar{u}''$, using (2), and then divide through by $\bar{u} + \bar{u}'\cot\theta$ (which will be proved in §7 never to vanish). From (2) and (5) we derive

$$ln\bar{p}' = \lambda(\bar{u} + \bar{u}'\cot\theta), \text{ where } \lambda = 2\bar{u}'/\{(\gamma-1)(c^2 - \bar{u}^2) - (\gamma+1)\bar{u}'^2\} \tag{40}$$

and the differential equation for x_n becomes

$$\left.\begin{aligned} &x_n'' + Bx_n' + Cx_n - n^2D(x_n + d_nT(\theta)/(\gamma-1)) = 0 \\ &\text{where } B = \cot\theta + \lambda\{2\bar{u} + 3\bar{u}'\cot\theta + (\gamma+1)\bar{u}'ln\bar{p}'\} \\ &\qquad C = 2 + \lambda\{\bar{u}' + (\gamma-1)\bar{u}ln\bar{p}'\} \\ &\text{and } D = (1 + \lambda\bar{u}')\operatorname{cosec}^2\theta \end{aligned}\right\} \tag{41}$$

Now if we substitute in (20) the values of x_n, x_n' ($= y_n$) and η_n given by (29), (31) and (33), we obtain

$$a_n = d_n\bar{p}\cot\theta/(\gamma-1)\bar{\rho}(\bar{u} + \bar{u}'\cot\theta)^2 \text{ evaluated with } \theta = \theta_w \tag{42}$$

and thus (29) and (31) can be written:

$$\left.\begin{aligned} &\text{When } \theta = \theta_w, \\ &x_n = -d_n\bar{p}/(\gamma - 1)\bar{p}(\bar{u} + \bar{u}'\cot\theta) \\ &\text{and } x_n' = 2d_n\bar{p}(2\bar{u} - \bar{u}'\cot\theta)\cot\theta/(\gamma^2 - 1)\bar{p}(\bar{u} + \bar{u}'\cot\theta) \end{aligned}\right\} \tag{43}$$

Equations (41) and (43) evidently suffice to determine the function $x_n/d_n = f_n(\theta)$, say, uniquely. "In general" this function will *not* satisfy

$$f_n'(\theta_s) = 0 \tag{44}$$

(We postpone till §7 the consideration of the "exceptional case" in which (44) is satisfied.) Consequently the requirement

$$\left.\begin{aligned} d_n f_n'(\theta) &= 2\epsilon\bar{u} \text{ if } n = 1 \\ &= 0 \text{ otherwise} \end{aligned}\right\} \text{ when } \theta = \theta_s \tag{45}$$

(from (35)) determines d_1 as a known multiple of ϵ; and this determines $x_1 = d_1 f_1(\theta)$, and then y_1, z_1, ξ_1, η_1 and α_1. Similarly if $n \neq 1$ we obtain $d_n = 0$, so that x_n (and thus also y_n, z_n, ξ_n, η_n and α_n) will be identically 0. Finally, since the capital letters X_n, Y_n, etc. satisfy the same equations—except that $Y_n = 0$ when $\theta = \theta_s$ even when $n = 1$—we see that the capital letters are all identically 0 in the "general" case.

Write $x_1/\epsilon = x$, $y_1/\epsilon = y$, etc.; and finally $\alpha_1/\epsilon = \alpha$, $d_1/\epsilon = d$. Thus the new quantities, $x, y, \cdots, d$ will be independent of ϵ, and $x/d = x_1/d_1 = f_1(\theta)$, determining them. Equations (9) reduce, of course, to: $u = \bar{u} + \epsilon x\cos\phi$, $v = \bar{v} + \epsilon y\cos\phi$, etc. Finally the shock-wave equation (10) reduces to $\theta = \theta_w + \epsilon\alpha\cos\phi$. Comparison with (34) shows that this represents a right circular cone, of semi-angle θ_w, but having a "yaw" δ given by

$$\delta/\epsilon = \alpha \tag{46}$$

(The "yaw" of this conical surface means the angle between its axis of symmetry and the direction of motion of the undisturbed flow.) As is to be expected from symmetry, the axis of symmetry of the shock-wave cone lies in the plane $\phi = 0$.

7. Validity and Uniqueness. The experimental confirmation (§8) is sufficient justification of the theory for practical purposes; but it would still be of mathematical interest to prove that the solution of the problem is unique, and that the present method does give a valid approximation to it. A *completely* rigorous proof of this seems to be very difficult, because of (a) the non-linearity of the fundamental differential equations, and (b) the peculiar nature of the boundary conditions.[8] If however we make the plausible assumption that the variables $u, v, \cdots$, and their spacial derivatives depend *continuously* on the parameter ϵ,[9]

[8] In fact, the "uniqueness" theorem is, strictly speaking, false, as the work of Taylor and Maccoll shows in the non-yaw case.

[9] A study of the second approximation shows that this dependence is *not* everywhere analytic.

the proof becomes easy. Write $\omega(\epsilon)$ to denote any quantity which $\to 0$ uniformly with ϵ, and $q_n = \max\{|x_n|, |y_n|, \cdots, |\eta_n|\}$. Then the equations derived above become *exact* if we add $q_n\omega(\epsilon)$ to their right-hand sides (note that the only operations performed on them were integrations and algebraic manipulations). Thus, since one easily shows that q_n/x_n is bounded (as ϵ varies), we have that the "exact" value of x_n/d_n, say g_n, is determined by the same equations as f_n ((41) and (43)), except that the coefficient C in (41) must be replaced by $C + \omega_1(\epsilon)$, and the values of g_n and g_n' at θ_w are respectively $f_n + \omega_2(\epsilon)$ and $f_n' + \omega_3(\epsilon)$. Hence $g_n = f_n + \omega(\epsilon)$. If, as assumed hitherto, $f_n'(\theta_s) \neq 0$, then the "exact" form of (45)—viz. $d_n g_n'(\theta_s) = 2\epsilon u(1 + \omega(\epsilon))$ if $n = 1$, and $= \omega(\epsilon)$ otherwise—shows that the "exact" values of x_n, etc., differ from the values found above by quantities of the form $\epsilon\omega(\epsilon)$—i.e., by $o(\epsilon)$ at most.

These differences are probably of order $O(\epsilon^2)$, though this seems harder to *prove*; and in practice the errors in the theory would be best determined by carrying the approximation a step further, and evaluating the terms in ϵ^2.

It remains to discuss whether the "exceptional case" in which (44) holds, can occur. (If it could, the preceding justification would break down; and in fact the present approximation would no longer determine the flow, since if (44) holds with $n = 1$ the present theory would be inconsistent, and if (44) holds with $n \neq 1$, d_n and so x_n and, similarly, X_n, would be indeterminate to the present approximation.) We first show that the "exceptional case" can occur for at most a few values of n. In fact, suppose (44) holds; we assert that

$$n^2 \leqq \{\sup_{\theta_s \leqq \theta \leqq \theta_w} C/D\}/\{1 + \sup_{\theta_s \leqq \theta \leqq \theta_w} T(\theta)/(\gamma - 1)f_n(\theta_w)\} \tag{47}$$

(It will be shown below that $D > 0$ and that $-(\gamma - 1)f_n(\theta_w) > \sup_{\theta_s \leqq \theta \leqq \theta_w} T(\theta) > 0$.)

For suppose not. Let θ_0 be the largest value of θ for which $f_n'(\theta) = 0$; thus, from (43), $\theta_s \leqq \theta_0 < \theta_w$ and $f_n'(\theta) > 0$ in (θ_0, θ_w), and so $-f_n(\theta_0) > -f_n(\theta_w) > T(\theta_0)/(\gamma - 1)$. Hence (41) shows that $f_n''(\theta_0) < 0$, and so $f_n'(\theta) < 0$ for all $\theta > \theta_0$ sufficiently close to θ_0—a contradiction.

From (47) we can deduce a numerical bound for "exceptional" values of n, as follows.

From the non-yaw solution we have that $\bar{u}' < 0$ if $\theta > \theta_s$. Now (2) can be written

$$\{(\bar{u} + \bar{u}'\cot\theta)\sin^2\theta\}' = -\bar{u}'ln\bar{\rho}'\sin\theta\cos\theta \tag{48}$$

If $ln\bar{\rho}' = 0$ for some $\theta > \theta_s$, we may suppose θ_1 to be the largest such θ; from (8), we have $\theta_1 < \theta_w$ and $ln\bar{\rho}' < 0$ in (θ_1, θ_w). Thus (48) shows that, when $\theta = \theta_1$, $(\bar{u} + \bar{u}'\cot\theta)\sin^2\theta > (\bar{u}_w + \bar{u}_w'\cot\theta_w)\sin^2\theta_w > 0$ (on using (7) and (8)); here the suffixes w denote that the corresponding quantities are to be evaluated with $\theta = \theta_w$. But (2) and (5) show that, when $\theta = \theta_1$, $\bar{u} + \bar{u}'' = 0 = \bar{u}'' + \bar{u}'\cot\theta + 2\bar{u}$, and so $\bar{u} + \bar{u}'\cot\theta = 0$—a contradiction. This shows that $ln\bar{\rho}' < 0$ for all θ in (θ_s, θ_w), and thus also that $\bar{u} + \bar{u}'\cot\theta > 0$.

It readily follows that $\lambda\bar{u}' \geqq 0$, and thus that $D > 0$ (in fact, $D > 1$).

Direct differentiation now shows that $(-ln\bar{p}')\tan\theta$ is an increasing function of θ in (θ_s, θ_w); hence, from (8),

$$0 \leqq -ln\bar{p}' \leqq 2\cot\theta/(\gamma+1) \tag{49}$$

Suppose now that sup $T(\theta)$ occurs when $\theta = \theta^*$; thus either $\theta^* = \theta_s$ or $T'(\theta^*) = 0$,[10] and in either case (37) and (39) give

$$\sup T(\theta) = \bar{p}/\bar{\rho}\bar{u} \text{ for } \theta = \theta^*$$

$$< \bar{p}/\bar{\rho}\bar{u}_w$$

$$< (\bar{p}_w/\bar{\rho}_w\bar{u}_w)\beta^{(\gamma-1)/(\gamma+1)} \text{ where } \beta = \sin^2\theta_w \operatorname{cosec}^2\theta^*,$$

on using (3) and (49).
Hence, from (43), (7) and (8),

$$(\sup T(\theta))/(\gamma-1)\,|f_n(\theta_w)| < \beta^{(\gamma-1)/(\gamma+1)}$$

$$\cdot\left\{1-\left(\frac{2a^2}{U^2\sin^2\theta_w}+(\gamma-1)\right)/(\gamma+1)\right\} < \frac{2}{\gamma+1}\beta^{(\gamma-1)/(\gamma+1)} \tag{50}$$

Again, one can verify by direct differentiation, on using (49), that

$$\{ln(\bar{p}/\bar{\rho}(\bar{u}+\bar{u}'\cot\theta))\}' > 4\cot\theta/(\gamma+1)$$

and so, if $\theta_s \leqq \theta \leqq \theta_w$,

$$\bar{p}_w/\bar{\rho}_w(\bar{u}_w+\bar{u}'_w\cot\theta_w) > (\sin\theta_w\operatorname{cosec}\theta)^{4/(\gamma+1)}\bar{p}/\bar{\rho}(\bar{u}+\bar{u}'\cot\theta).$$

Hence sup $T(\theta) = \bar{p}/\bar{\rho}\bar{u}$ for $\theta = \theta^*$

$$\leqq \bar{p}/\bar{\rho}(\bar{u}+\bar{u}'\cot\theta) < \beta^{-2/(\gamma+1)}\bar{p}_w/\bar{\rho}_w(\bar{u}_w+\bar{u}'_w\cot\theta_w),$$

and so

$$(\sup T(\theta))/(\gamma-1)\,|f_n(\theta_w)| < \beta^{-2/(\gamma+1)} \tag{51}$$

On combining (50) and (51) we readily obtain (since $\beta > 1$)

$$(\sup T(\theta))/(\gamma-1)\,|f_n(\theta_w)| < \{2/(\gamma+1)\}^{2/(\gamma+1)} < 0.86 \tag{52}$$

And it is easy to see that

$$C/D < (4\sin^2\theta_w + 2(\gamma-1))/(\gamma+1) < 2 \tag{53}$$

On inserting these estimates in (47), we see that the "exceptional" case can occur at most for $0 \leqq n \leqq 3$.

On using the results of the actual computations, we obtain from (47) a bound which is actually less than 1 in the range arising in practice. Thus the "exceptional case" can occur in practice at most for $n = 0$. The case $n = 0$ could presumably be eliminated also; but there seems to be no way of confirming this without more trouble than the point is worth.

[10] Both cases can occur.

8. Numerical Results and Applications. (i) *The flow in a typical case.* The values of all the quantities x, y, etc., in a typical case, are given in Table 1[11], together with the values of the coefficients in (41) (for $n = 1$), and of the fundamental non-yaw quantities.

(*Added in proof.* When $n = 1$ one can "solve" (41), expressing x/d in terms of the non-yaw quantities by means of quadratures. But this does not materially simplify the computations, which were actually carried out from (41) directly.)

(ii) *The normal head-force coefficient* As was pointed out in §1, the preceding theory should apply with reasonable exactness to an actual conical-headed projectile in flight, in the neighbourhood of the conical head The pressure at the point $(r, \theta_s + \epsilon\cos\phi, \phi)$ of the head will be $p = \bar{p}_s + \epsilon\eta_s\cos\phi$, since $\bar{p}_s' = 0$ (from

TABLE 1

Non-yaw Quantities, Coefficients, and Complete Solution for the case $\theta_s = 15°$, $U/a = 1.901$

$\theta_s = 35.43°$,	$c/a = 2.924$,	$d = 0.00730$,	$a = 0.377$

θ	$\bar{u}/c$	$-\bar{w}'/c$	$\bar{p}/\rho_1 l$	$\bar{\rho}/\rho_1$	B	$D-C$	$\frac{D,T(\theta)}{c(\gamma-1)}$	$-z/c$	$\frac{\gamma}{c} = \frac{x'}{c}$	$-100l/c$	z/c	$\xi/\rho_1 l$	$\eta/\bar{p}$
θ_s	.5294	.3217	1.250	1.171	−7.88	6.82	0.0	.021	.478	.00	.036	1.52	1.84
35°	.5318	.3148	1.263	1.180	−4.23	6.21	0.1	.024	.457	.01	.042	1.46	1.75
33°	.5423	.2853	1.310	1.211	0.18	4.74	0.4	.040	.440	.03	.073	1.39	1.62
31°	.5518	.2569	1.347	1.235	1.07	4.26	0.6	.056	.457	.06	.107	1.41	1.61
29°	.5602	.2287	1.380	1.256	1.42	4.19	0.9	.072	.486	.09	.147	1.46	1.64
27°	.5677	.2001	1.409	1.275	1.63	4.38	1.2	.090	.525	.12	.194	1.52	1.68
25°	.5742	.1709	1.435	1.292	1.82	4.80	1.6	.109	.575	.16	.254	1.58	1.73
23°	.5800	.1408	1.459	1.307	2.01	5.45	2.1	.130	.634	.20	.327	1.63	1.76
21°	.5840	.1093	1.481	1.321	2.25	6.41	2.9	.153	.713	.25	.420	1.65	1.77
19°	.5873	.0759	1.500	1.334	2.57	7.79	4.2	.180	.819	.31	.543	1.67	1.76
17°	.5893	.0400	1.514	1.342	3.02	9.82	6.4	.211	.962	.39	.707	1.62	1.71
15°	.5900	.0000	1.521	1.347	3.73	12.9	5.9	.248	1.18	.29	.948	1.47	1.54

(1) and (6)); the suffixes s denote that the quantities are to be evaluated with $\theta = \theta_s$. From this it is easy to evaluate the resultant aerodynamic forces on the projectile head. Let D be the diameter of the base of this cone (D = the "calibre"), and let l be its perpendicular height ($= \frac{1}{2}D\cot\theta_s$). Then it is found that the head forces reduce to:

(a) a component $\pi\bar{p}_s D^2/4$ along the axis of symmetry (the "axial drag")[12], and
(b) a perpendicular force of magnitude $K_{NH}\rho_1 D^2 U^2\epsilon$, where

$$K_{NH} = \pi\eta_s\cot\theta_s/8\rho_1 U^2, \tag{54}$$

[11] Based on computations carried out by the Ballistic Research Laboratory, Aberdeen Proving Ground, under I. E. Segal. More accurate results have since been obtained by the Computing Section, Center of Analysis, Massachusetts Institute of Technology, Technical Report No. 3, (1947).

[12] As is clear from symmetry, the axial drag is, to the present approximation, unchanged by yaw. The effect of the yaw is considerable in practice, but is of order ϵ^2.

acting through a point on the axis of symmetry at a distance h_H from the vertex, where

$$h_H = (2l \sec^2 \theta_s)/3 \tag{55}$$

From (20) and (4) we have

$$\eta_s = -\bar{\rho}_s\{\bar{u}_s x_s + d(c^2 - \bar{u}_s^2)/2\gamma\} \tag{56}$$

enabling K_{NH} (the "normal head force coefficient") to be determined. The behaviour of K_{NH}, computed for various values of θ_s and Mach number, is illustrated by Fig. 2[13]. It is found that the limiting values of K_{NH}, as $U \to \infty$, are 0.7759, 0.7051, and 0.5972 for $\theta_s = 10°$, 20° and 30° respectively.

One can derive rough estimates for the forces on the complete projectile, and thence for the conditions for the yaw to remain small in flight, which are in rough agreement with the known facts; however, an accurate theoretical deter-

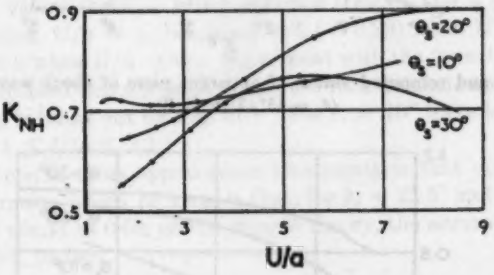

FIG. 2. Normal Head Force Coefficient(K_{NH})

mination of these forces would involve the difficult, though not hopeless, problem of determining the flow in the neighbourhood of the cylindrical portion of the projectile.

(iii) *The yaw of the shock-wave.* A more accurate test of the present theory is the fact that it determines the ratio of the yaw of the shock-wave to the yaw of the projectile (equations 46 and 42). Spark photographs may be obtained which give in effect the projections of the conical head and shock-wave on the plane of the photographic plate. Let θ_w^* denote the "apparent" value of θ_w—i.e., half the angle between the two lines which form the image of the shock-wave, and let δ^* denote the "apparent" value of δ—i.e., the angle between the axis of symmetry of the shock-wave image and the direction of motion. Similarly let θ_s^* and ϵ^* denote the "apparent" values of θ_s and ϵ. If the photographic plate is parallel to the direction of motion, one readily obtains (on neglecting ϵ^2) that $\theta_s^* = \theta_s$, $\theta_w^* = \theta_w$, and $\delta^*/\epsilon^* = \delta/\epsilon = \alpha$, giving an observational check on the theory. In Fig. 3, observed values of δ^* and ϵ^* are plotted, and may be compared with the theoretical line. (The values are for $\theta_s = 15°$ and $U/a = 2$ approximately.)

[13] Figures 2 and 4 are based on computations of the Center of Analysis, M.I.T.; their complete tabulations have now been published. (See footnote 11).

The departures are all within observational error.[14] Figure 4[13] gives the behaviour of the ratio δ/ϵ computed for various Mach numbers and cone semi-angles. The limiting values for $U \to \infty$ are respectively 1.0570, 1.0642 and 1.0798 for $\theta_s = 10°, 20°$ and $30°$.

Another test for the theory is that θ_w^* should be independent of ϵ. This also has been verified to hold within experimental error, but various difficulties make this a less accurate test than the foregoing.

(iv) *The second non-yaw régime.* One interesting feature of the computations (not shown in Figs. 2 and 4) is that, when carried out for the *second* of the two

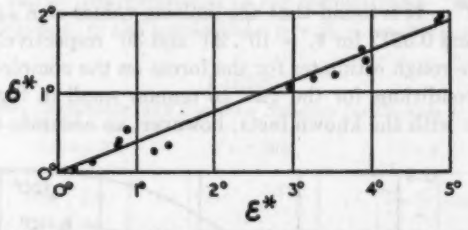

FIG. 3. Observed and computed values of apparent yaws of shock wave and projectile ($\theta_s = 15°$, $U/a \fallingdotseq 2$)

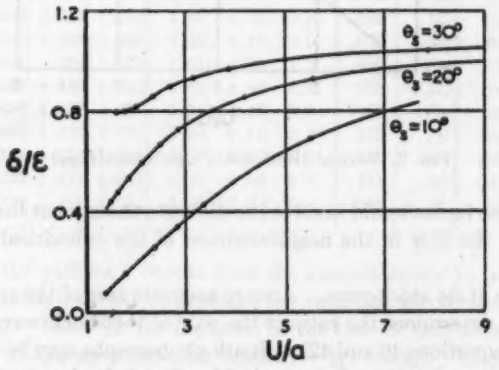

FIG. 4. Ratio of shock-wave yaw to yaw of projectile

theoretically possible non-yaw régimes, they lead to much larger values for K_{NB} and δ/ϵ. This provides a partial explanation of the puzzling fact that this second régime is not observed in ballistic practice; for in practice ϵ would be large enough to lead to an impossibly large value of δ. It would seem that the second régime, though perhaps stable for small enough yaws ϵ, is unstable for the yaws usually present in practice.

[14] The data for Fig. 3 are provided by the courtesy of the Ballistic Research Laboratory, Aberdeen Proving Ground. Each dot in the figure is the mean between two readings of the same plate by different observers. These two readings differ by more than their distance from the theoretical line.

(v) *Comparison with approximate theories.* Various approximate treatments have been devised[15], which are applicable to head shapes more general than the conical; hence it is desirable to check them in the conical case. Such methods usually involve disregarding the non-uniformity of the entropy and the non-irrotationality of the flow. The former of these approximations amounts to taking p/ρ^γ = constant, which leads to $d = 0$ in (19); thus the relative error is of the order of magnitude of $d\bar{p}/\eta$. As is known, this error is small if the shock is not very strong, as will usually be the case in practice (for example, in the case considered in Table 1, $d\bar{p}/\eta < 0.005$), but which will fail for large Mach numbers. As for the assumption of irrotationality, we find curl $\boldsymbol{v} = (\epsilon/r) \sin \phi \operatorname{cosec} \theta(t', -t, 0)$, in which the first component becomes infinite at θ_s. However, the assumption really involves the neglect of only $\boldsymbol{v} \times$ curl $\boldsymbol{v}$, which is of the order of $\epsilon d\bar{p}/\bar{p}$. Thus both these approximations should be valid in the usual ballistic range, but not for large Mach numbers.

The simplest approximate method is due to Tsien[15]; it assumes that θ_s is small, and neglects the shock-wave entirely. Tsien obtains $h_H = 2l/3$ (compare (55)), and values for K_{NH} which decrease steadily (for a given θ_s) from $\pi/4$ (= 0.79) when $U/a = 1$, becoming $\pi/8$ (= 0.39) when $U/a = \operatorname{cosec} \theta_s$, and having limit 0 when $U/a \to \infty$. Agreement with the "exact" values found above is to be expected only for small θ_s and for Mach numbers not much greater than 1; and this is borne out by Fig. 2.[16] (For $\theta_s = 10°$ there is agreement to within 10% for $1 \leqq U/a \leqq 2.3$.)

The most refined of these approximate treatments is that of Sauer[15]. The numerical illustration which he gives is that, for $\theta_s = 23.5°$ and $U/a = 2.378$, the value of δ/ϵ would be 0.68; on the present theory, the accurate value would be 0.71, about 5% higher.

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[15] See C. Ferrari, "Determination of the Pressure on a Solid of Revolution with a Pointed Nose in Supersonic Yawing Motion", Atti della R. Academia delle Scienze di Torino 1936–7; H. S. Tsien, "Supersonic Flow over an Inclined Body of Revolution", Journal of Aeronautical Sciences 1938, pp. 480–483; and R. Sauer, "Ueberschallströmung um beliebig geformte Geschossspitzen unter kleinem Anstellwinkel", Luftfahrtforschung 19, pp. 148–152. See also T. v. Karman and Moore, "Resistance of Slender Bodies Moving with Supersonic Velocities", Transactions of the American Society of Mechanical Engineers, vol. 54 (1932).

[16] For a more detailed comparison of Tsien's theory with the present one, see Z. Kopal, "Remarks on Limitations of Linearized Theory in Supersonic Flow", Center of Analysis, M.I.T., Technical Report No. 2 (1947).

ON THE DETERMINATION OF THE EIGENFUNCTIONS OF FREDHOLM EQUATIONS

By G. F. Carrier

It is well-known that certain iteration procedures will lead to the determination of the eigenvalues and eigenfunctions of the homogeneous Fredholm equation with symmetric kernel.[1] A simple extension is presented here which improves the convergence of such iterations so that the number of integrations to be performed for a certain degree of accuracy is markedly reduced. The method is analogous to one in matrix algebra.[2]

We consider the equation

$$\int_a^b K(x, t)\varphi(t)\, dt = K\cdot\varphi = \varphi(x)/\lambda. \tag{1}$$

The first equality above defines the meaning of $(K\cdot)$. It is known that

$$\lim_{n\to\infty} (K^{n+1}\cdot f)/(K^n\cdot f) = 1/\lambda_1,$$

where λ_1 is the smallest eigenvalue of (1) and f is an arbitrary integrable function. Furthermore,

$$1/\lambda_1 \simeq \int_a^b (K^{n+1}\cdot f)(K^n f)\, dx \Big/ \int_a^b (K^n\cdot f)^2\, dx \tag{2}$$

is a good approximation even for moderate values of n. One may also normalize $K^{n+1}\cdot f$ as an approximation to φ_1, the eigenfunction associated with λ_1. Similarly, for the higher eigenvalues one may write analogous equations for $\lambda_1\lambda_2$, etc.

In order to improve the approximation to λ_1 and φ_1 without increasing n, one needs only to use the rough values already obtained for $\lambda_2, \lambda_3, \cdots$ (or approximations to these eigenvalues known in advance). Let us call these approximations $\lambda_2^*, \lambda_3^*, \cdots$; then, e.g., the polynomial operator $P(K\cdot) = K\cdot(K\cdot - 1/\lambda_2^*)(K\cdot - 1/\lambda_3^*)$ will, in general, provide a much better approximation to φ_1 when one uses $P(K\cdot)f$ instead of $K^3\cdot f$. This is seen when one writes

$$P = P(K\cdot)f = \sum \alpha_j P(\lambda_j)\varphi_j \tag{3}$$

where the α_j are the generalized Fourier coefficients of f. Similarly

$$\lambda_1 \simeq \int_a^b (K\cdot P)P\, dx \Big/ \int_a^b P^2\, dx \tag{4}$$

will, in general, be better than the results given by Eq. (2) when the highest power of $(K\cdot)$ in Eqs. (2) and (4) are the same.

[1] See e.g., K. Hohenemser, *Praktische Wege zur angenaeherten Schwingungsberechnung elastischer System*, Ingenieur Archiv I, 271 (1930).

[2] W. M. Kincaid, *Numerical methods for finding characteristic roots and vectors of matrices*, To appear in Quart. of Applied. Math.

A simple example: One may readily demonstrate the advantages of this method by a trivial example. We consider the kernel

$$\pi K(x, t) = \begin{cases} (\pi - x)t, & 0 \leq t \leq x \\ (\pi - t)x, & x \leq t \leq \pi \end{cases}. \tag{5}$$

The solutions of (1) are, as is well known,

$$\varphi_n = \sin nx; \qquad \lambda_n = 1/n^2.$$

If we choose $f(x)$ to be the constant unity, we find

$$f_1 = K \cdot f = x(\pi - x)/2,$$

$$f_2 = K^2 \cdot f = (\pi^3 x - 2\pi x^3 + x^4)/24.$$

The eigenvalue obtained from

$$\lambda_1 \simeq \int f_1 f_2 \, dx \Big/ \int f_1^2 \, dx \tag{6}$$

is .9988 whereas, [for $P = P(K \cdot) f = K \cdot (K \cdot - .1) f$]

$$\lambda_1 P(\lambda_1) \simeq \int P^2 \, dx \Big/ \int f_1 P \, dx, \quad \text{or} \quad \lambda_1 \simeq 1.0001. \tag{7}$$

Note that the "guess" for λ_2^* is 1/10. This value is chosen because the second eigenvalue (.25) cannot influence the result on account of the symmetry of f. Also note that the number of applications of $(K \cdot)$ is the same in both of the foregoing computations.

If we also compute $\varphi_1(\pi/4)/\varphi_1(\pi/2)$ using f_2 as an approximation to the non-normalized φ_1, we obtain $f_2(\pi/4)/f_2(\pi/2) = .7125$. With $P(K \cdot)f$ as an approximation to φ_1, we find $P(\pi/4)/P(\pi/2) = .7085$. The precise value is, of course, .7071. Thus we see that the use of polynomials in $(K \cdot)$ can reduce the number of iterations required to obtain a given accuracy either in the eigenvalue determination or in finding the eigenfunctions.

It is evident (compare Eqs. (4) and (7)) that there is considerable freedom of choice in choosing the polynomial and the products thereof from which λ_1 is deduced. Any known facts as to symmetry, etc., may obviously be used to advantage.

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TABLE OF ROOTS FOR NATURAL FREQUENCIES IN COAXIAL TYPE CAVITIES

H. B. Dwight

In the calculation of natural frequencies for the various modes of oscillation in a resonant cavity of the coaxial type, numerical values of roots are required for the following equations:

$$J_n(x)N_n(kx) - J_n(kx)N_n(x) = 0 \tag{1}$$

and

$$J_n'(x)N_n'(kx) - J_n'(kx)N_n'(x) = 0. \tag{2}$$

where $J_n(x)$ and $N_n(x)$ are Bessel functions of the first and second kinds.

Asymptotic expressions for most of the roots were given by J. McMahon[1] for both Eqs. (1) and (2).

It was found by D. O. North[2] that there is a set of roots of Eq.(2), smaller than the others, which is not given by the asymptotic expressions. Curves showing values of these roots were published by R. Truell.[2] These roots are the first roots, that is, the smallest. These smaller roots do not occur for Eq.(1), and when $n = 0$ they do not occur for Eq.(2).

A table of roots of Eq.(1) was published by A. Kalähne[3] and in the various editions of "Tables of Functions" by Jahnke and Emde[4] since 1909. A short table was given by A. N. Lowan and A. Hillman.[5] Numerical values of the roots of Eq.(2), which in communication work is often of equal importance with Eq.(1), do not seem to have been published up to this time.

It may be noted that values of x_0' are not tabulated, since $x_0' = x_1$.

None of the tabulated values in this paper correspond to the dominant mode of oscillation of a coaxial line, for which $x = 0$.

The computation of the values in the following table was done mainly by students in 1942 at Massachusetts Institute of Technology under a grant from the A. E. Kennelly Fund. The values of the smaller roots, not given by the asymptotic expressions, were calculated by the Computation Section of the Research Laboratory of Electronics at M.I.T. The work was originally suggested by W. L. Barrow.

REFERENCES

1. J. McMahon, Annals of Math., **9**, p. 23, (1894).
2. R. Truell, Jour. of Applied Physics, **14**, p. 350, (1943).
3. A. Kalähne, Zeitschr. für Math. u. Phys., **54**, pp. 55-86, (1907).
4. "Tables of Functions" by E. Jahnke and F. Emde.
5. A. N. Lowan and A. Hillman, Jour. Math. and Phys. **22**, p. 203, (1943).

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Roots, $x_n^{(s)}$, of $J_n(x)N_n(kx) - J_n(kx)N_n(x) = 0$

k	1ST ROOT $(k-1)\,x_0^{(1)}$	2ND ROOT $(k-1)\,x_0^{(2)}$	3RD ROOT $(k-1)\,x_0^{(3)}$	4TH ROOT $(k-1)\,x_0^{(4)}$	5TH ROOT $(k-1)\,x_0^{(5)}$
1.0	3.1416	6.2832	9.4248	12.5664	15.7080
1.1	3.1412	6.2830	9.4247	12.5663	15.7079
1.2	3.1403	6.2825	9.4243	12.5660	15.7077
1.3	3.1389	6.2818	9.4239	12.5657	15.7074
1.4	3.1371	6.2809	9.4233	12.5652	15.7071
1.5	3.1351	6.2799	9.4226	12.5647	15.7066
1.6	3.1329	6.2787	9.4218	12.5641	15.7062
1.8	3.128	6.276	9.420	12.563	15.705
2.0	3.123	6.273	9.418	12.561	15.704
2.5	3.110	6.266	9.413	12.558	15.701
3.0	3.097	6.258	9.408	12.553	15.697
3.5	3.085	6.250	9.402	12.549	15.694
4.0	3.073	6.243	9.396	12.545	15.690
5	3.053	6.228	9.39		
6	3.035	6.215	9.38		
8	3.006	6.191	9.36		
10	2.983	6.172	9.34		
12	2.963	6.155	9.32		
14	2.947	6.140	9.31		
16	2.934	6.126			
18	2.922	6.114			
20	2.911	6.104			
25	2.89	6.08			
30	2.87	6.06			
35	2.86	6.05			
40	2.85	6.03			
45	2.84	6.02			
50	2.83	6.02			

k	1ST ROOT $(k-1)\,x_1^{(1)}$	2ND ROOT $(k-1)\,x_1^{(2)}$	3RD ROOT $(k-1)\,x_1^{(3)}$	4TH ROOT $(k-1)\,x_1^{(4)}$	5TH ROOT $(k-1)\,x_1^{(5)}$
1.0	3.1416	6.2832	9.4248	12.5664	15.7080
1.1	3.1427	6.2837	9.4251	12.5666	15.7082
1.2	3.1455	6.2852	9.4261	12.5674	15.7088
1.3	3.1498	6.2873	9.4275	12.5684	15.7096
1.4	3.1550	6.2900	9.4293	12.5698	15.7107
1.5	3.1609	6.2931	9.4314	12.5713	15.7119
1.6	3.1675	6.2965	9.4337	12.5731	15.7133
1.8	3.182	6.304	9.439	12.577	15.716
2.0	3.197	6.312	9.444	12.581	15.720
2.5	3.235	6.335	9.460	12.593	15.729
3.0	3.271	6.357	9.476	12.605	15.739
3.5	3.305	6.381	9.493	12.619	15.750
4.0	3.336	6.403	9.509	12.631	15.760
5	3.389	6.445	9.541	12.657	15.782
6	3.432	6.482	9.572	12.682	15.802
8	3.498	6.546	9.626	12.728	15.842
10	3.547	6.598	9.673	12.770	15.879
12	3.583	6.639	9.714	12.807	15.913
14	3.611	6.674	9.749	12.840	15.943
16	3.634	6.704	9.780	12.870	15.971
18	3.652	6.728	9.806	12.896	15.997
20	3.667	6.749	9.830	12.920	16.020
25	3.696	6.790	9.88	12.97	16.07
30	3.717	6.820	9.91	13.01	16.11
35	3.732	6.844	9.94	13.04	16.14
40	3.743	6.861	9.96	13.06	16.17
45	3.752	6.875	9.98	13.09	16.19
50	3.760	6.887	9.99	13.10	16.21

k	1ST ROOT $(k-1)\,x_1^{(1)}$	2ND ROOT $(k-1)\,x_1^{(2)}$	3RD ROOT $(k-1)\,x_1^{(3)}$	4TH ROOT $(k-1)\,x_1^{(4)}$	5TH ROOT $(k-1)\,x_1^{(5)}$
1.0	3.1416	6.2832	9.4248	12.5664	15.7080
1.1	3.1470	6.2859	9.4266	12.5677	15.7090
1.2	3.1613	6.2931	9.4314	12.5713	15.7119
1.3	3.1822	6.3038	9.4385	12.5767	15.7162
1.4	3.208	6.3170	9.4474	12.5834	15.7216
1.5	3.237	6.3324	9.4578	12.5912	15.7278
1.6	3.27	6.349	9.4693	12.5998	15.7348
1.8	3.36	6.387	9.495	12.619	15.750
2.0	3.4	6.43	9.523	12.640	15.767

k	1ST ROOT $(k-1)\,x_2^{(1)}$	2ND ROOT $(k-1)\,x_2^{(2)}$	3RD ROOT $(k-1)\,x_2^{(3)}$	4TH ROOT $(k-1)\,x_2^{(4)}$	5TH ROOT $(k-1)\,x_2^{(5)}$
1.0	3.1416	6.2832	9.4248	12.5664	15.7080
1.1	3.1542	6.2895	9.4290	12.5695	15.7105
1.2	3.1874	6.3063	9.4402	12.5780	15.7172
1.3	3.236	6.3311	9.4568	12.5904	15.7272
1.4	3.294	6.3619	9.4776	12.6061	15.7397
1.5	3.36	6.397	9.5016	12.6242	15.7543
1.6	3.43	6.437	9.5283	12.6443	15.7704
1.8	3.6	6.523	9.587	12.689	15.806
2.0	3.7	6.62	9.652	12.738	15.846
2.5		6.9	9.83	12.874	15.956
3.0			10.0	13.02	16.07
3.5			10.2	13.2	16.19
4.0				13.3	16.3

k	1ST ROOT $x_1'^{(1)}$	2ND ROOT $(k-1)x_1'^{(2)}$	3RD ROOT $(k-1)\,x_1'^{(3)}$	4TH ROOT $(k-1)\,x_1'^{(4)}$	5TH ROOT $(k-1)\,x_1'^{(5)}$	6TH ROOT $(k-1)\,x_1'^{(6)}$
1.0	1.	3.1416	6.2832	9.4248	12.5664	15.7080
1.1	.953	3.1441	6.2845	9.4256	12.5670	15.7085
1.2	.910	3.1509	6.2878	9.4279	12.5687	15.7098
1.3	.872	3.1609	6.2928	9.4312	12.5712	15.7118
1.4	.837	3.174	6.2991	9.4354	12.5743	15.7143
1.5	.805	3.188	6.3064	9.4403	12.5780	15.7172
1.6	.776	3.205	6.3146	9.4457	12.5820	15.7205
1.8	.723	3.241	6.333	9.458	12.591	15.728
2.0	.677	3.282	6.353	9.471	12.601	15.736
2.5	.585	3.396	6.410	9.509	12.629	15.758
3.0	.514	3.516	6.472	9.550	12.660	15.783
3.5	.457	3.636	6.538	9.593	12.692	15.808
4.0	.411	3.753	6.606	9.638	12.725	15.83
5	.341	3.969	6.746	9.732	12.79	15.89
6	.290	4.153	6.887	9.829		
8	.223	4.433	7.152	10.028		
10	.180	4.623	7.379	10.223		
12	.151	4.754	7.562	10.403		
14	.130	4.848	7.707	10.561		
16	.114	4.917	7.820	10.697		
18	.102	4.970	7.910	10.811		
20	.092	5.011	7.982	10.908		
25	.073	5.08	8.11	11.09		
30	.061	5.13	8.19	11.20		
35	.053	5.16	8.25	11.29		
40		5.18	8.29	11.35		
45		5.20	8.32			
50		5.22	8.34			

k	1ST ROOT $x_2'^{(1)}$	2ND ROOT $(k-1)x_2'^{(2)}$	3RD ROOT $(k-1)\,x_2'^{(3)}$	4TH ROOT $(k-1)\,x_2'^{(4)}$	5TH ROOT $(k-1)\,x_2'^{(5)}$	6TH ROOT $(k-1)\,x_2'^{(6)}$
1.0	2.	3.1416	6.2832	9.4248	12.5664	15.7080
1.1	1.905	3.1485	6.2866	9.4271	12.5681	15.7093
1.2	1.821	3.1669	6.2958	9.4332	12.5727	15.7130
1.3	1.744	3.1942	6.3094	9.4422	12.5795	15.7184
1.4	1.673	3.229	6.3265	9.4536	12.5880	15.7252
1.5	1.608	3.27	6.3464	9.4668	12.5979	15.7332
1.6	1.548	3.32	6.369	9.4816	12.6089	15.7420
1.8	1.438	3.4	6.419	9.515	12.6337	15.7618
2.0	1.341	3.5	6.47	9.552	12.6612	15.7837
2.5	1.137		6.6	9.665	12.739	15.845
3.0	.977		6.8	9.77	12.82	15.912
3.5	.852		7.0	9.89	12.91	15.982
4.0	.752			10.0	13.0	16.06

k	1ST ROOT $x_3'^{(1)}$	2ND ROOT $(k-1)x_3'^{(2)}$	3RD ROOT $(k-1)\,x_3'^{(3)}$	4TH ROOT $(k-1)\,x_3'^{(4)}$	5TH ROOT $(k-1)\,x_3'^{(5)}$	6TH ROOT $(k-1)\,x_3'^{(6)}$
1.0	3.	3.1416	6.2832	9.4248	12.5664	15.7080
1.1	2.858	3.1557	6.2902	9.4295	12.5699	15.7108
1.2	2.731	3.1933	6.3090	9.4420	12.5793	15.7183
1.3	2.614	3.2489	6.3370	9.4606	12.5932	15.7294
1.4	2.507	3.319	6.3718	9.4839	12.6107	15.7434
1.5	2.407	3.40	6.4125	9.5110	12.6310	15.7597
1.6	2.312	3.49	6.458	9.5412	12.6537	15.7778
1.8	2.138	3.7	6.56	9.609	12.7044	15.8184
2.0	1.979		6.67	9.684	12.761	15.863
2.5	1.643		7.0	9.90	12.92	15.99
3.0	1.388			10.1	13.09	16.13
3.5	1.196				13.3	16.27
4.0	1.048				13.5	16.4

SECOND APPROXIMATION TO THE LAMINAR BOUNDARY LAYER FLOW OVER A FLAT PLATE*

HENRY LEONARD ALDEN

Introduction. The problem of flow of an incompressible fluid of small viscosity was first formulated by L. Prandtl with the resultant well known boundary layer theory. In essence, the theory of the boundary layer is the first approximation of the asymptotic solution of the Navier-Stokes differential equations for large Reynolds numbers [1]. This observation naturally leads to the question of second and higher approximations in the asymptotic expansion of the exact solution. It is the purpose of the present paper to consider these questions for the simplest case: the boundary layer over a flat plate. The first approximation to this problem was given by H. Blasius [2], Topfer [3], S. Goldstein [4], and H. L. Dryden [5]. The results of the present calculations show appreciable deviations from the known solution for points near the leading edge of the plate when higher order terms are considered.

1. Basic Equations. The orientation of the plate and coordinate axes with respect to the flow is shown in figure (1). The plate lies along the x-axis at zero angle of attack with its leading edge at the origin, and extends to infinity in the positive direction. (r, θ) are the corresponding polar coordinates, and (u, v) are the horizontal and vertical velocity components.

As $r \to \infty$ everywhere outside the boundary layer, $u \to U$ and $v \to 0$. The fluid flow is subjected to no external forces and is assumed to be continuous, incompressible, and two-dimensional. With these conditions in mind one may eliminate the pressure terms in the Navier-Stokes equations of motion by cross-differentiation and subtraction, and thus obtain the following equation for the stream function:

$$\frac{\partial \psi}{\partial y}\frac{\partial \nabla^2 \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \nabla^2 \psi}{\partial y} = \nu \nabla^2 \nabla^2 \psi \tag{1}$$

$$\partial\psi/\partial y = u \qquad \partial\psi/\partial x = -v \qquad \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$$

where ψ is the stream function. The method of solution of (1) will be the expansion of ψ in a power series of the parameter ν. Previous investigation has shown[1] that the power series in cartesian coordinates reduces to

$$\psi = (\nu U x)^{1/2}\{f_1(\xi) + (\nu/Ux)f_2(\xi) + (\nu/Ux)^2 f_3(\xi) \cdots\} \tag{2}$$

where $\xi = \frac{1}{2}(U/\nu x)^{1/2} y$.

The functions $f_n(\xi)$ would be determined by substituting ψ into (1), identifying the powers of ν and solving the resulting equations. However, the powers of

* The material in this paper is taken from a thesis done in the spring term, 1947, at the Massachusetts Institute of Technology under the direction of Dr. H. S. Tsien.

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ν/Ux introduce a line singularity of infinite length along the y-axis, and this in turn causes difficulty when the boundary conditions are applied for $r \to \infty$.

In order to avoid this difficulty the coordinate system will be transformed from cartesian to parabolic, and the calculations carried out in terms of the new variables.

2. Transformation of Coordinates. Equation (1) is to be transformed into parabolic coordinates defined as follows (see figure 2):

$$z = re^{i\theta} = x + iy = (\sigma + i\tau)^2 = \zeta^2 \tag{3}$$

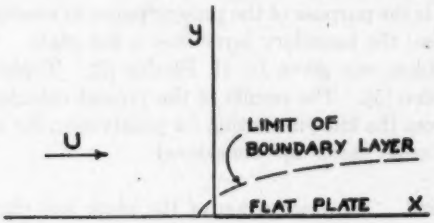

FIG. 1.

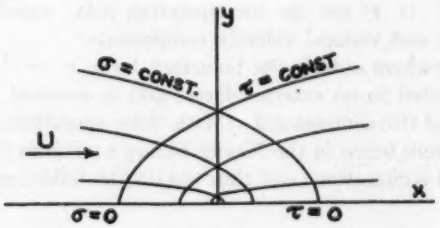

FIG. 2.

This furnishes the relationships

$$\begin{cases} \sigma = r^{1/2}\cos\tfrac{1}{2}\theta, \qquad \tau = r^{1/2}\sin\tfrac{1}{2}\theta, \qquad r = \sigma^2 + \tau^2 \\ |\, d\zeta/dz\,| = \tfrac{1}{2}r^{-1/2} = M = \tfrac{1}{2}(\sigma^2 + \tau^2)^{-1/2} \end{cases} \tag{4}$$

Since the transformation is conformal, (1) can be written immediately as

$$\frac{\partial\psi}{\partial\tau}\frac{\partial}{\partial\sigma}(M^2\nabla^2\psi) - \frac{\partial\psi}{\partial\sigma}\frac{\partial}{\partial\tau}(M^2\nabla^2\psi) = \nu\nabla^2(M^2\nabla^2\psi)$$

where $\nabla^2 = \partial^2/\partial\sigma^2 + \partial^2/\partial\tau^2$.

Carrying out the indicated differentiations and multiplying by $4(\sigma^2 + \tau^2)^2$ one obtains equation (1) in parabolic coordinates:

$$(\sigma^2+\tau^2)\left\{\frac{\partial\psi}{\partial\tau}\frac{\partial}{\partial\sigma}\nabla^2\psi-\frac{\partial\psi}{\partial\sigma}\frac{\partial}{\partial\tau}\nabla^2\psi\right\}-2\nabla^2\psi\left\{\sigma\frac{\partial\psi}{\partial\tau}-\tau\frac{\partial\psi}{\partial\sigma}\right\}$$
$$=\nu\left\{4-4\left(\sigma\frac{\partial}{\partial\sigma}+\tau\frac{\partial}{\partial\tau}\right)+(\sigma^2+\tau^2)\nabla^2\right\}\nabla^2\psi \tag{5}$$

3. Expansion of the Stream Function. In order to apply the method of series expansion it is necessary that the differential equation (5) be expressed in terms of two independent coordinates which, in the regions where the parameter ν is important, are of equal order of magnitude. It will be observed that for points within a thin boundary layer the value of θ is very small, and hence that (4) reduces to $\sigma = x^{\frac{1}{2}}$ and $\tau = \frac{1}{2}y/x^{\frac{1}{2}}$. It is known that under these conditions x and $y/\nu^{\frac{1}{2}}$ are of the same order of magnitude, and hence σ and $\tau/\nu^{\frac{1}{2}}$ are of the same order. For the purposes of calculation, then, the coordinates σ and $\eta = \tau/\nu^{\frac{1}{2}}$ will be employed. The stream function is now written in the form of an infinite power series in ν:

$$\psi = \nu^{\frac{1}{2}}\psi_1(\sigma,\eta)+\nu^{3/2}\psi_2(\sigma,\eta)+\nu^{5/2}\psi_3(\sigma,\eta)\cdots \tag{6}$$

With this form all the derivatives appearing in (5) can be expressed as series:

$$\partial\psi/\partial\sigma = \nu^{1/2}\partial\psi_1/\partial\sigma+\nu^{3/2}\partial\psi_2/\partial\sigma\cdots$$
$$\partial\psi/\partial\tau = \quad\partial\psi_1/\partial\eta+\quad\nu\partial\psi_2/\partial\eta\cdots\text{ etc.}$$

Remembering that $\tau = \nu^{1/2}\eta$ and making the proper substitutions in (5), one is then able to identify each coefficient of ν and set up equations for $\psi_1, \psi_2, \cdots \psi_n$. The lowest power of ν is $(-\frac{1}{2})$, and its coefficient gives

$$\sigma^2\frac{\partial\psi_1}{\partial\eta}\frac{\partial^3\psi_1}{\partial\eta^2\partial\sigma}-\sigma^2\frac{\partial\psi_1}{\partial\sigma}\frac{\partial^3\psi_1}{\partial\eta^3}-2\sigma\frac{\partial^2\psi_1}{\partial\eta^2}\frac{\partial\psi_1}{\partial\eta}=\sigma^2\frac{\partial^4\psi_1}{\partial\eta^4} \tag{7}$$

The next power of ν is $(\frac{1}{2})$ and its coefficient gives

$$\sigma^2\left\{\frac{\partial\psi_1}{\partial\eta}\frac{\partial^3\psi_2}{\partial\sigma\partial\eta^2}+\frac{\partial\psi_1}{\partial\eta}\frac{\partial^3\psi_1}{\partial\sigma^3}+\frac{\partial\psi_2}{\partial\eta}\frac{\partial^3\psi_1}{\partial\eta^2\partial\sigma}-\frac{\partial\psi_1}{\partial\sigma}\frac{\partial^3\psi_2}{\partial\eta^3}-\frac{\partial\psi_1}{\partial\sigma}\frac{\partial^3\psi_1}{\partial\eta\partial\sigma^2}-\frac{\partial\psi_2}{\partial\sigma}\frac{\partial^3\psi_1}{\partial\eta^3}\right\}$$
$$+\eta^2\left\{\frac{\partial\psi_1}{\partial\eta}\frac{\partial^3\psi_1}{\partial\eta^2\partial\sigma}-\frac{\partial\psi_1}{\partial\sigma}\frac{\partial^3\psi_1}{\partial\eta^3}\right\}-2\sigma\left\{\frac{\partial\psi_1}{\partial\eta}\frac{\partial^2\psi_2}{\partial\eta^2}+\frac{\partial\psi_1}{\partial\eta}\frac{\partial^2\psi_1}{\partial\sigma^2}+\frac{\partial\psi_2}{\partial\eta}\frac{\partial^2\psi_1}{\partial\eta^2}\right\}$$
$$+2\eta\left\{\frac{\partial\psi_1}{\partial\sigma}\frac{\partial^2\psi_1}{\partial\eta^2}\right\}=4\frac{\partial^2\psi_1}{\partial\eta^2}-4\eta\frac{\partial^3\psi_1}{\partial\eta^3}-4\sigma\frac{\partial^3\psi_1}{\partial\sigma\partial\eta^2}$$
$$+\sigma^2\left\{2\frac{\partial^4\psi_1}{\partial\eta^2\partial\sigma^2}+\frac{\partial^4\psi_2}{\partial\eta^4}\right\}+\eta^2\frac{\partial^4\psi_1}{\partial\eta^4} \tag{8}$$

The equation governing the third term may be obtained in a similar manner, but here the discussion will be restricted to the first two terms as given by (7) and (8).

4. Reduction to Ordinary Equations. By using the proper arrangement of parabolic variables, (7) and (8) can be reduced to ordinary equations. For this purpose the stream function terms are written in the form

$$\psi_1 = \sqrt{U}\sigma f_1(\sqrt{U}\eta) = \sqrt{U}\sigma f_1(\xi)$$

$$\psi_2 = \frac{1}{\sqrt{U}\sigma} f_2(\xi), \qquad \psi_n = \frac{1}{(\sqrt{U}\sigma)^{2n-3}} f_n(\xi)$$

where $\xi = \sqrt{U}\eta = \sqrt{U/\nu}\tau$.

With these expressions the stream function can be recast into a power series in $\nu/U\sigma^2$:

$$\psi = \sigma\sqrt{U\nu}\{f_1 + (\nu/U\sigma^2)f_2 + (\nu/U\sigma^2)^2 f_3 \cdots\} \tag{9}$$

It will be noticed that $U\sigma^2/\nu$ is very nearly equal the position Reynolds number Ux/ν for points close to the plate, so that for this region (9) is virtually an expansion in reciprocal powers of the Reynolds number.

The derivatives of ψ_1 and ψ_2 are now obtained and substituted in (7) and (8) giving respectively

$$f_1'''' + f_1 f_1''' + f_1' f_1'' = 0, \tag{10}$$

$$\begin{aligned} f_2'''' + f_1 f_2''' + 3f_1' f_2'' + f_1'' f_2' - f_1''' f_2 \\ = -\xi^2 f_1'''' - \xi^2 f_1 f_1''' + 4\xi f_1''' + \xi^2 f_1' f_1'' + 2\xi f_1 f_1'', \end{aligned} \tag{11}$$

where $(')$ indicates differentiation with respect to ξ. The determination of the first two stream function terms is reduced to solving (10) and (11).

5. Boundary Conditions. The first derivatives of ψ with respect to the parabolic variables are

$$\partial\psi/\partial\sigma = (\partial\psi/\partial y)(\partial y/\partial\sigma) + (\partial\psi/\partial x)(\partial x/\partial\sigma)$$

$$\partial\psi/\partial\tau = (\partial\psi/\partial y)(\partial\psi/\partial\tau) + (\partial\psi/\partial x)(\partial x/\partial\tau)$$

These reduce to

$$\partial\psi/\partial\sigma = 2\tau u - 2\sigma v$$

$$\partial\psi/\partial\tau = 2\sigma u + 2\tau v$$

On the plate $u = v = 0$, so that

$$(\partial\psi/\partial\sigma)_{\tau=0} = 0 \qquad (\partial\psi/\partial\tau)_{\tau=0} = 0$$

As $\tau \to \infty$, $u \to U$, $v \to 0$, and so for any fixed value of σ

$$(\partial\psi/\partial\sigma)_{\tau=\infty} \to O(2\tau U).$$

Similarly as $\sigma \to \infty$ for a fixed value of τ outside the boundary layer,

$$(\partial\psi/\partial\tau)_{\sigma\to\infty} \to O(2\sigma U).$$

From (9) one obtains

$$\partial\psi/\partial\sigma = \sqrt{U\nu}\{f_1 - (\nu/U\sigma^2)f_2 \cdots\},$$
$$\partial\psi/\partial\tau = \sigma U\{f_1' + (\nu/U\sigma^2)f_2' \cdots\}.$$

Applying the conditions for $\tau = \xi = 0$ one finds immediately that

$$f_1(0) = f_2(0) = f_n(0) = 0,$$
$$f_1'(0) = f_2'(0) = f_n'(0) = 0.$$

Examination of $\partial\psi/\partial\sigma$ for $\tau \to \infty$ shows that the conditions are satisfied by $f_1 \to 2\xi - \beta$ (β = constant), and $f_2 = f_n \to 0$. In like manner, for points outside the boundary layer, the conditions on $\partial\psi/\partial\tau$ as $\sigma \to \infty$ are satisfied by $f_1' \to 2$ which checks with f_1.

6. Solution of the First and Second Approximations. Referring to (10) it is seen that one direct integration is possible with respect to ξ:

$$f_1''' + f_1 f_1'' = 0 \tag{12}$$

The constant of integration is zero since $f_1''' \to 0$ and $f_1 f_1'' \to 0$ as $\xi \to \infty$. This is equivalent to the original equation given by Prandtl, and has been solved by numerical integration for the boundary conditions prescribed here.[4]

In view of (12) the right side of (11) may be simplified to:

$$f_2'''' + f_1 f_2''' + 3f_1' f_2'' + f_1'' f_2' - f_1''' f_2 = 2\xi^2 f_1' f_1'' - 2\xi f_1 f_1''.$$

In this form the equation can be integrated once to give

$$f_2''' + f_1 f_2'' + 2f_1' f_2' - f_1'' f_2 = f_1^2 + \xi^2 f_1'^2 - 2\xi f_1 f_1' + C_1 \tag{13}$$
$$= F(\xi) + C_1.$$

Multiplying by the integrating factor f_1 and using (12), one may perform the second integration and obtain:

$$f_1 f_2'' + (f_1^2 - f_1')f_2' + f_1'' f_2 = \int_0^\xi f_1(F + C_1)\, d\xi + C_2. \tag{14}$$

Next it is noted that $f_2 = f_1'$ is a solution of the homogeneous part of this equation, thus indicating that one may assume a solution in the form $f_2 = f_1' g(\xi)$. Making the substitution and dividing by $f_1 f_1'$ one obtains

$$g'' + \left(2\frac{f_1''}{f_1'} + f_1 - \frac{f_1'}{f_1}\right)g' = \frac{1}{f_1 f_1'}\left(\int_0^\xi f_1(F + C_1)\, d\xi + C_2\right).$$

Solving for g' by using the integrating factor $f_1'^2/f_1f_1''$ gives

$$g' = \frac{f_1f_1''}{f_1'^2}\left\{\int_0^\xi \frac{f_1'}{f_1'f_1''}\left(\int_0^\xi f_1(F + C_1)\,d\xi + C_2\right)d\xi + C_3\right\}.$$

Since $f_2 = f_1'g$, one obtains finally

$$(15)\qquad f_2 = f_1'\int_0^\xi \frac{f_1f_1''}{f_1'^2}\left\{\int_0^\xi \frac{f_1'}{f_1'f_1''}\left(\int_0^\xi f_1(F + C_1)\,d\xi + C_2\right)d\xi + C_3\right\}d\xi + C_4f_1'.$$

In order to check (15) for improper integrals in the neighborhood of $\xi = 0$ one may employ the series expansion of f_1 which is of the form[4] $f_1 = k_1\xi^2 + k_2\xi^5$. Upon substitution of this into (15) it becomes evident that the integrands remain finite as $\xi \to 0$, providing $C_2 = 0$. Recalling the boundary conditions for $\xi = 0$, one observes that equation (14) determines the value of C_2, which is zero.

Direct differentiation of (15) will now yield the following constants for f_2 and its derivatives at $\xi = 0$.

$$f_2(0) = 0, \qquad f_2''(0) = C_3f_1''(0),$$
$$f_2'(0) = C_4f_1''(0), \qquad f_2'''(0) = C_1.$$

It follows that $C_4 = 0$. The formula for f_2 may be further simplified by performing the indicated integration involving the constant C_3:

$$f_1'\int_0^\xi \frac{f_1f_1''}{f_1'^2}C_3\,d\xi = C_3f_1'\left[\xi - \frac{f_1}{f_1'}\right]_0^\xi = C_3(\xi f_1' - f_1).$$

Thus f_2 becomes

$$f_2 = C_3(\xi f_1' - f_1) + f_1'\int_0^\xi \frac{f_1f_1''}{f_1'^2}\left\{\int_0^\xi \frac{f_1'}{f_1'f_1''}\left(\int_0^\xi f_1(F + C_1)\,d\xi\right)d\xi\right\}d\xi.$$

For purposes apparent later this will be abbreviated as

$$(16)\qquad f_2 = C_3(\xi f_1' - f_1) + \Delta f_2.$$

Formula (16) is the exact expression for f_2 with two of the integration constants eliminated by application of the boundary conditions at the surface of the plate. For the determination of the two remaining constants it will be necessary to find the behavior of f_2 as $\xi \to \infty$.

For large values of ξ, f_1 and its derivatives approach the following values[6]:

$$f_1 \to 2\xi - \beta, \qquad f_1' \to 2, \qquad f_1'' \to 4\gamma e^{-1/4(2\xi-\beta)^2}$$

where β & γ = constants and $\beta = 1.7207$.
These are to be substituted into (16) in order to study f_2 at large values of ξ. For this purpose one makes use of the first term of the asymptotic integration

$$\int e^{1/4(2\xi-\beta)^2}\,d\xi = \frac{e^{1/4(2\xi-\beta)^2}}{(2\xi - \beta)}\left\{1 + \frac{2}{(2\xi - \beta)^2} \cdots\right\}$$

Carrying out the substitution one finds that as $\xi \to \infty$,

$$f_2 \to \left(\frac{\beta^2 + C_1}{4}\right)\xi + (\beta C_3 + a) + H.O.T.$$

where "a" is a constant involving C_1. To satisfy the requirement that $f_2 \to 0$ as $\xi \to \infty$, C_1 must equal $-\beta^2$ and C_3 must be picked so that $\beta C_3 + a$ is zero. However, the values of all the constants grouped together in "a" are not known, thus preventing a predetermination of C_3.

7. Evaluation of the Second Stream Function Approximation. Although (16) is an explicit formula for f_2, the labor involved in a triple step by step integration is greater than that in an integration by standard methods[7] of the linear third order equation (13). Referring to (16) one observes that either f_2 or Δf_2 will satisfy (13), and that, by virtue of linearity, there is a complete set of initial conditions available for Δf_2:

$$\xi = 0; \quad \Delta f_2(0) = 0, \quad \Delta f_2''(0) = 0,$$
$$\Delta f_2'(0) = 0, \quad \Delta f_2'''(0) = C_1 = -\beta^2.$$

With these values the numerical integration of (13) to obtain Δf_2 can be started at the origin. This procedure was carried out as far as $\xi = 4.10$ in steps of approximately .09 units. At this stage it was found that Δf_2 tended toward its limiting value quite slowly but that certain terms effectively dropped out of the equation, thus allowing a simple solution.

Using the asymptotic values of f_1 in equation (13) and neglecting negative exponentials for large values of ξ, one may write

$$\Delta f_2''' + (2\xi - \beta)\Delta f_2'' + 4\Delta f_2' = 0$$

For values of $\xi = 3.9$ and larger this equation was found correct to four significant figures. Integrating gives

$$\Delta f_2'' + (2\xi - \beta)\Delta f_2' + 2\Delta f_2 = 2a \tag{17}$$

where "a" is a constant. Integrating again,

$$\Delta f_2' + (2\xi - \beta)\Delta f_2 = a(2\xi - \beta) + b \tag{18}$$

where "b" is the second constant of integration. Integration a third time using the standard integrating factor for first order equations gives:

$$\Delta f_2 = a + be^{-1/4(2\xi-\beta)^2} \int^{\xi} e^{1/4(2\xi-\beta)^2} d\xi + ce^{-1/4(2\xi-\beta)^2}$$

Since negative exponentials have been neglected it will be assumed that the third term on the right is small enough to drop out. The limit of the second term as $\xi \to \infty$ is zero, so that Δf_2 approaches the limit "a". Therefore,

$$f_2 \to C_3\beta + a = 0; \qquad C_3 = -\beta/a$$

TABLE I

X	ξ	f_1	f_1'	f_1''	f_2	f_2'	f_2''
0	0	0	0	1.3282	0	0	2.6479
.1	0.0910	0.0055	0.1208	1.3280	0.0106	0.2285	2.3767
.2	0.1819	0.0220	0.2416	1.3265	0.0409	0.4321	2.0965
.3	0.2729	0.0494	0.3621	1.3223	0.0885	0.6096	1.7998
.4	0.3639	0.0879	0.4821	1.3142	0.1507	0.7589	1.4820
.5	0.4549	0.1371	0.6010	1.3009	0.2256	0.8784	1.1416
.6	0.5458	0.1972	0.7185	1.2814	0.3097	0.9660	0.7804
.7	0.6368	0.2678	0.8340	1.2546	0.4004	1.0201	0.4032
.8	0.7278	0.3488	0.9466	1.2200	0.4943	1.0392	0.0195
.9	0.8188	0.4399	1.0557	1.1771	0.5884	1.0237	−0.3597
1.0	0.9097	0.5408	1.1605	1.1259	0.6794	0.9743	−0.7198
1.1	1.0007	0.6509	1.2603	1.0665	0.7647	0.8938	−1.0465
1.2	1.0917	0.7699	1.3543	0.9998	0.8414	0.7854	−1.3238
1.3	1.1826	0.8971	1.4420	0.9269	0.9071	0.6547	−1.5383
1.4	1.2736	1.0320	1.5228	0.8491	0.9601	0.5157	−1.6802
1.5	1.3646	1.1740	1.5964	0.7681	0.9990	0.3514	−1.7437
1.6	1.4556	1.3223	1.6625	0.6856	1.0238	0.1928	−1.7284
1.7	1.5465	1.4762	1.7212	0.6037	1.0343	0.0390	−1.6391
1.8	1.6375	1.6352	1.7724	0.5241	1.0311	−0.1036	−1.4862
1.9	1.7285	1.7985	1.8166	0.4483	1.0159	−0.2299	−1.2837
2.0	1.8194	1.9655	1.8542	0.3778	0.9899	−0.3362	−1.0485
2.1	1.9104	2.1357	1.8855	0.3135	0.9551	−0.4202	−0.7966
2.2	2.0014	2.3084	1.9114	0.2561	0.9141	−0.4813	−0.5482
2.3	2.0924	2.4833	1.9324	0.2060	0.8684	−0.5204	−0.3143
2.4	2.1833	2.6599	1.9491	0.1630	0.8200	−0.5393	−0.1071
2.5	2.2743	2.8378	1.9622	0.1269	0.7708	−0.5408	0.0669
2.6	2.3653	3.0168	1.9724	0.0973	0.7221	−0.5282	0.2037
2.7	2.4562	3.1966	1.9801	0.0733	0.6750	−0.5050	0.3031
2.8	2.5472	3.3770	1.9859	0.0544	0.6303	−0.4742	0.3683
2.9	2.6382	3.5579	1.9901	0.0397	0.5888	−0.4389	0.4033
3.0	2.7292	3.7391	1.9932	0.0285	0.5504	−0.4018	0.4138
3.1	2.8201	3.9205	1.9954	0.0201	0.5157	−0.3641	0.4053
3.2	2.9111	4.1021	1.9969	0.0140	0.4842	−0.3281	0.3838
3.3	3.0021	4.2838	1.9980	0.0095	0.4559	−0.2945	0.3539
3.4	3.0931	4.4656	1.9987	0.0064	0.4306	−0.2638	0.3198
3.5	3.1840	4.6474	1.9992	0.0042	0.4079	−0.2364	0.2845
3.6	3.2750	4.8293	1.9995	0.0028	0.3875	−0.2118	0.2503
3.7	3.3660	5.0112	1.9997	0.0018	0.3693	−0.1906	0.2185
3.8	3.4569	5.1931	1.9998	0.0011	0.3529	−0.1720	0.1895
3.9	3.5479	5.3751	1.9999	0.0007	0.3378	−0.1561	0.1642
4.0	3.6389	5.5570	1.9999	0.0004	0.3243	−0.1422	0.1422
4.1	3.7299	5.7389	2.0000	0.0003	0.3121	−0.1301	0.1235
4.2	3.8208	5.9209	2.0000	0.0002	0.3006	−0.1197	0.1077
4.3	3.9118	6.1028	2.0000	0.0001	0.2901	−0.1106	0.0944
4.4	4.0028	6.2848	2.0000	0.0001	0.2804	−0.1025	0.0834
4.5	4.0937	6.4667	2.0000	0.0000	0.2715	−0.0954	0.0738
∞	∞	∞	2	0	0	0	0

The value of f_1 and its derivative are taken from the calculations in Reference 5. The variable X is a convenient index for purposes of numerical integration and varies linearly with ξ.

Having obtained Δf_2 and its derivatives up to $\xi = 3.9$ and three steps beyond, it is possible to obtain "a" and "b" from (17) and (18). These values are $a = -3.341$ and $b = 1.660$, and hence $C_3 = 1.994$. Using this value the C_3-term is added to Δf_2 to complete the solution (Table I). For large values of ξ,

$$f_2 = C_3\beta + a + be^{-1/4(2\xi-\beta)^2}\int^{\xi} e^{1/4(2\xi-\beta)^2}\,d\xi$$

Introducing the asymptotic form of the integral this becomes

$$f_2 = \frac{b}{(2\xi-\beta)}\left\{1 + \frac{2}{(2\xi-\beta)^2} + \frac{12}{(2\xi-\beta)^4}\cdots\right\}$$

This formula gives an approximation of f_2 for values of ξ above the range covered by the numerical integration.

8. Calculation of the Pressure Coefficient. The two-dimensional Navier-Stokes equations may be transformed directly into parabolic coordinates, giving

$$(\sigma^2+\tau^2)\left\{\frac{\partial^2\psi}{\partial\sigma^2}\frac{\partial\psi}{\partial\tau} - \frac{\partial^2\psi}{\partial\tau\partial\sigma}\frac{\partial\psi}{\partial\sigma}\right\} + \tau\left\{\left(\frac{\partial\psi}{\partial\tau}\right)^2 + \left(\frac{\partial\psi}{\partial\sigma}\right)^2\right\} - 4(\sigma^2+\tau^2)\frac{1}{\rho}\frac{\partial p}{\partial\tau} = \nu\left\{-2\sigma\nabla^2\psi + (\sigma^2+\tau^2)\frac{\partial}{\partial\sigma}\nabla^2\psi\right\}. \tag{19}$$

and

$$(\sigma^2+\tau^2)\left\{\frac{\partial^2\psi}{\partial\tau^2}\frac{\partial\psi}{\partial\sigma} - \frac{\partial^2\psi}{\partial\tau\partial\sigma}\frac{\partial\psi}{\partial\tau}\right\} + \sigma\left\{\left(\frac{\partial\psi}{\partial\tau}\right)^2 + \left(\frac{\partial\psi}{\partial\sigma}\right)^2\right\} - 4(\sigma^2+\tau^2)\frac{1}{\rho}\frac{\partial p}{\partial\sigma} = \nu\left\{2\tau\nabla^2\psi - (\sigma^2+\tau^2)\frac{\partial}{\partial\tau}\nabla^2\psi\right\}. \tag{20}$$

The pressure will be given by a series of the form

$$p = p_1(\sigma_1\eta) + \nu p_2(\sigma_1\eta) + \nu^2 p_3(\sigma_1\eta)\cdots$$

A substitution of p and ψ in (19) and (20) will give the derivatives of p, from which the pressure terms may be determined at once. It is found that $p_1 = p^0 =$ free stream pressure. Evaluation of p_2 gives a pressure coefficient of

$$C_p = \frac{p-p^0}{\frac{1}{2}\rho U^2} = \frac{\nu}{2U\sigma^2}\left\{\xi f_1 f_1' + \xi f_1'' - f_1^2 + \frac{\beta^2}{2}\right\}. \tag{21}$$

9. Results of the Analysis. The quantities of interest are plotted in terms of the nondimensional coordinates Ux/ν and Uy/ν, which are proportional to the physical coordinates x and y. The conversion from parabolic coordinates is given by

$$Ux/\nu = (U\sigma^2/\nu) - \xi^2$$

$$Uy/\nu = 2\sqrt{U\sigma^2/\nu}\cdot\xi = 2\sqrt{(Ux/\nu) + \xi^2}\cdot\xi$$

In the velocity profile graphs the parameter λ is introduced which varies linearly with Uy/ν. The outer limit of the boundary layer is a function of ξ, and its

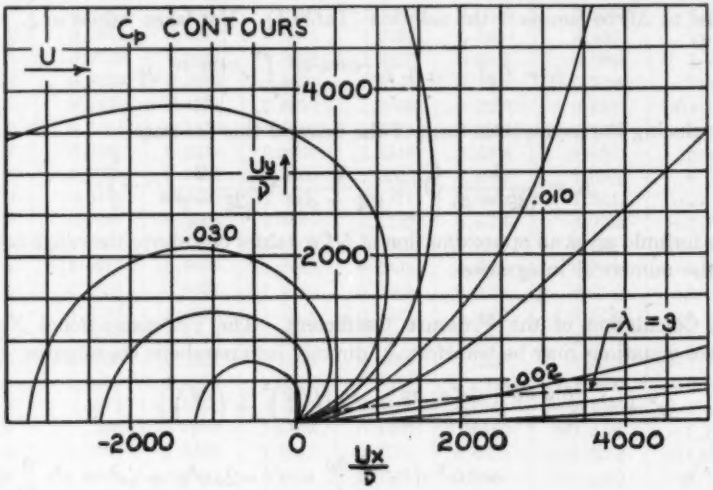

CHART I

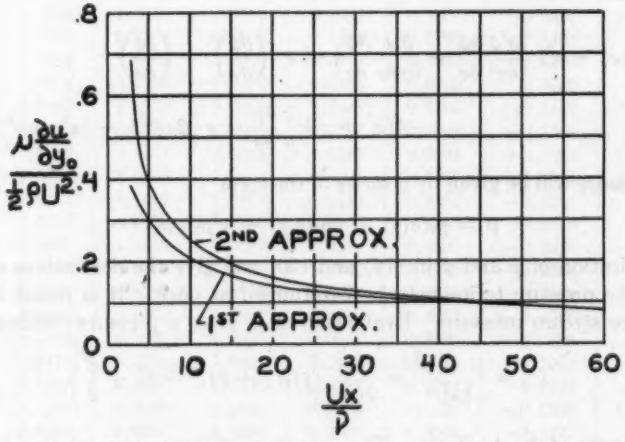

CHART II

position will be taken nominally at $\xi = 3$. λ is so scaled that it equals 3 when $\xi = 3$, and for large values of Ux/ν, $\lambda = \xi$. It is given by

$$\lambda = \frac{Uy/\nu}{2\sqrt{R_x + 9}}, \qquad R_x = \frac{Ux}{\nu}.$$

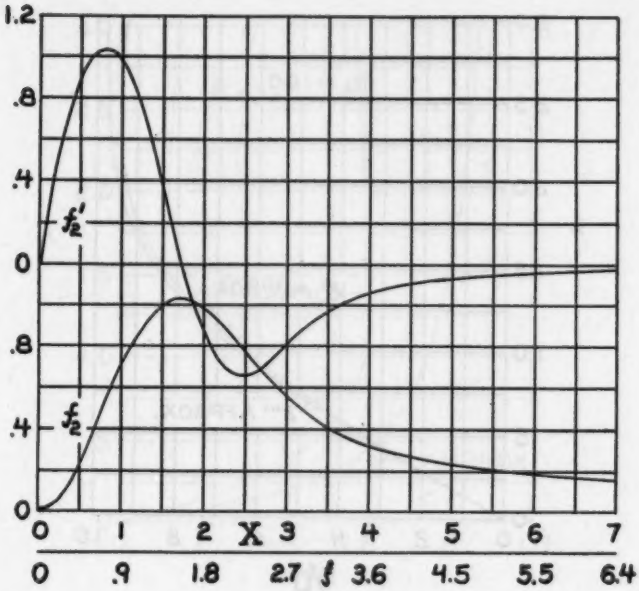

CHART III

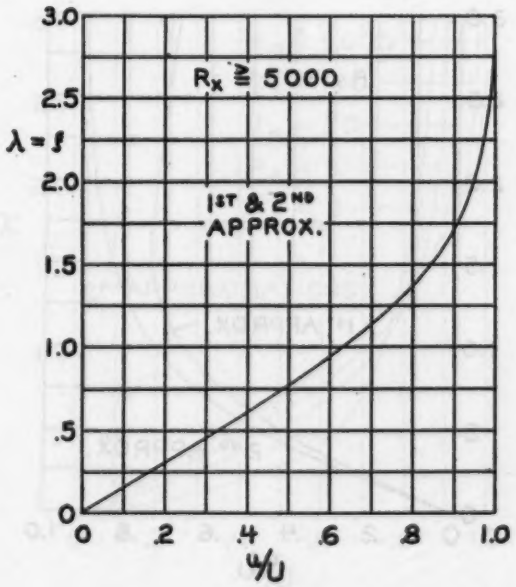

CHART IV

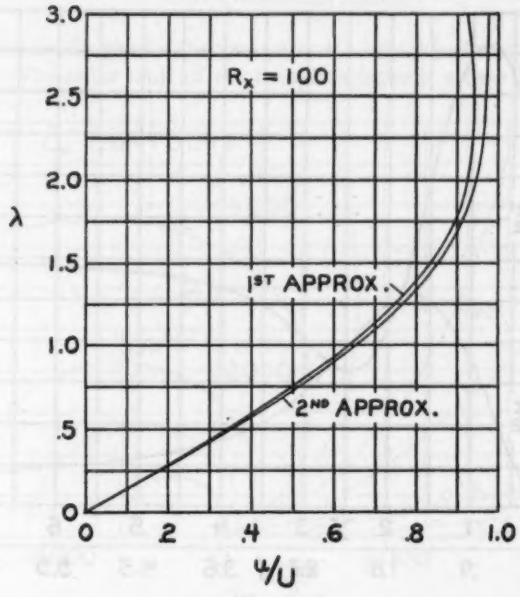

CHART V

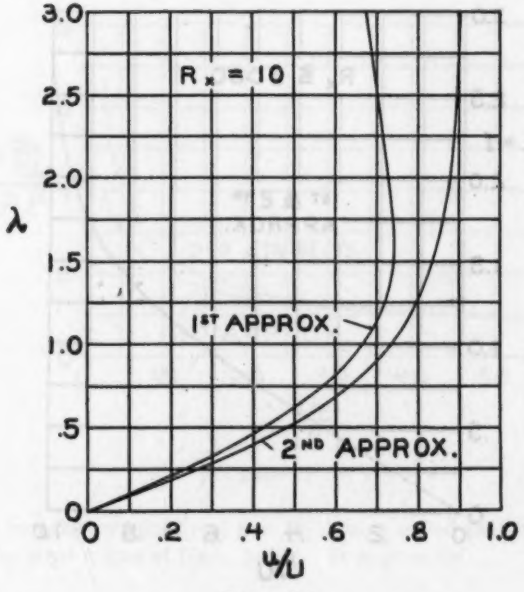

CHART VI

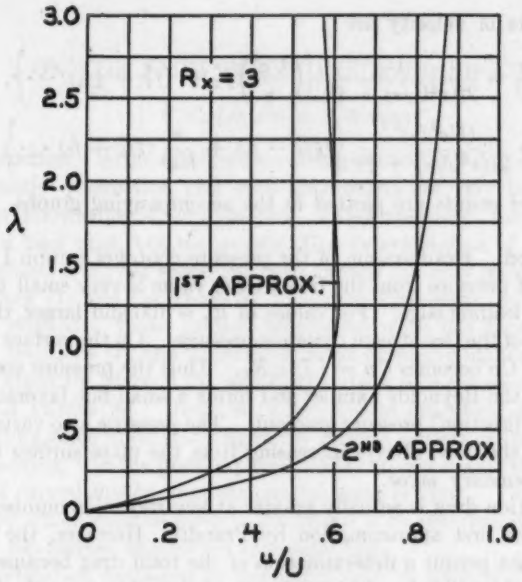

CHART VII

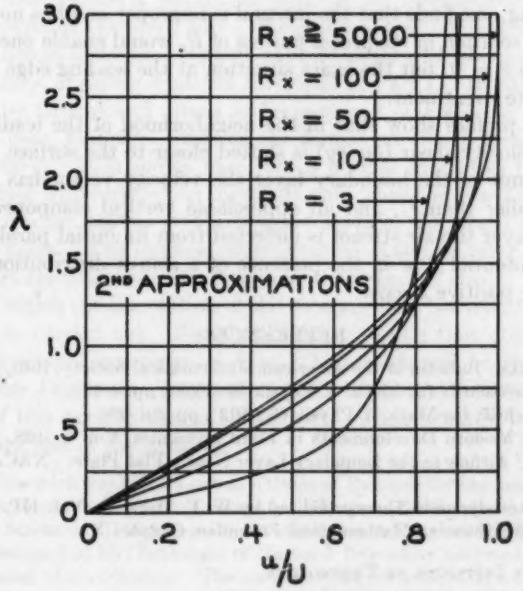

CHART VIII

The components of velocity are

$$\frac{u}{U} = \frac{1}{2((\sigma^2 U/\nu) + \xi^2)} \left\{ \left(\frac{U\sigma^2}{\nu} \right) f_1' + (f_2' + \xi f_1) \cdots \right\},$$

$$\frac{v}{U} = \frac{(U\sigma^2/\nu)^{1/2}}{2((\sigma^2 U/\nu) + \xi^2)} \left\{ (\xi f_1' - f_1) + \frac{\nu}{U\sigma^2} (\xi f_2' + f_2) \cdots \right\}.$$

These and other results are plotted in the accompanying graphs.

10. Discussion. Examination of the pressure contours (graph I) shows that the deviation of pressure from the free stream value is very small except in the region near the leading edge. For values of $R_x = 100$ and larger, this deviation is less than 1% of the free stream dynamic pressure. On the surface of the plate, the formula for Cp becomes $Cp = (.74)/R_x$. Thus the pressure coefficient falls off inversely as the Reynolds number and forms a small but favorable (decreasing in the flow direction) pressure gradient. The pressure also varies in a direction normal to the flow, always increasing from the plate surface to the outer limit of the boundary layer.

The skin friction drag is actually greater at low Reynolds numbers than that predicted by the first approximation by Prandtl. However, the calculations given here do not permit a determination of the total drag because of the pronounced effect of the third and higher terms of the solution at the leading edge of the plate. If one attempts to integrate the skin friction curve (graph II) to obtain total drag, one finds that the integral is improper and has no limit. The complete series solution in reciprocal powers of R_x would enable one to get valid results closer to $x = 0$; but the exact situation at the leading edge seems to require a separate treatment.

The velocity profiles show that in the neighborhood of the leading edge the zone of large velocity shear $(\partial u/\partial y)$ is shifted closer to the surface of the plate. At the outer limit of the boundary layer the velocity vector has a horizontal component smaller than U, and an appreciable vertical component. Beyond the boundary layer the air stream is deflected from its initial parallel flow, and behaves as a potential flow in the presence of a source distribution varying as $1/x^{1/2}$ along the positive x axis.

REFERENCES

1. Th. von Kármán: Bulletin of the American Mathematical Society, 1940, pp. 655–659.
2. H. Blasius: Zeitschrift fur Math. u. Physik 56 (1908), pp. 4–13.
3. Topfer: Zeitschrift fur Math. u. Physik 60 (1912), pp. 397–398.
4. S. Goldstein: Modern Developments in Fluid Dynamics, Vol. I, 1938, pp. 135–137.
5. H. L. Dryden: Airflow in the Boundary Layer Near a Flat Plate. NACA TR No. 562, 1936.
6. L. Prandtl: Aerodynamic Theory (Edited by W. F. Durand), Vol. III, pp. 84–90.
7. E. P. Adams: Smithsonian Mathematical Formulae, Chapter X.

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ON THE FLOW BEHIND CURVED SHOCKS[1]

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1. Introduction. Although flows involving curved and detached shocks occur very frequently in practice, and such phenomena are very important in drag studies, our knowledge is very limited. Experimentally, they have been observed for a long time, but the quantitative determination of the pressure distribution behind a curved shock has been carried out only recently by Professor R. Ladenburg [1] and his group at Princeton University by means of interferometric methods. Some limited theoretical work has been carried out by Crocco [2], Hentsche [3], and Thomas [4], some extensive numerical calculations[2] have been made by Maccoll [5], and finally there is the semi-empirical work of Laitone and Pardee [6]. However, a systematic analytical approach, with a view of getting some understanding of the general aspects of the problem, seems to be still lacking. In the present paper, we try to investigate the problem of the flow behind curved shocks from the general point of view. Yet concrete results are obtained which may be directly compared with experiments, and which are also useful for further theoretical investigations.

The investigation contained in this paper may be roughly divided into three parts. Part I (§§2–3) deals with general relations for flow behind curved shocks. These include a variational principle for iso-energetic flows (with vorticity and variable entropy), and a discussion of the differential relations of the first order across a shock. Part II (§§4–5) deals with shocks attached to the solid boundary, applying relations developed in Part I. In particular, relations developed by Crocco and others are reproduced as a specific case. The most interesting case, however, concerns the formation of a normal shock over the solid boundary.[3] It is shown here that a normal shock can be formed over a convex surface with continuous curvature only when the Mach number exceeds a certain supersonic value. For a concave surface, the Mach number must be below that value. The physical interpretation of this peculiar behaviour is also given. Part III deals with the flow behind a detached shock. A scheme is developed by which the calculation of the subsonic flow between the shock and the body may be carried out. Some preliminary results thus obtained agree very well with available experimental data. This work will be reported elsewhere by Mr. John Dugundji. In conclusion, it is pointed out that much further extension of this work should be carried out. These are now being investigated by some with one of the authors.

[1] Part of this work was carried out as a thesis of Rubinov for the master's degree (1947) when both authors were at Brown University. Further work done by Lin was partly sponsored by the Mechanics Division of the Naval Ordnance Laboratory.

[2] Prof. Emmons and Mr. Drebinger of Harvard University informed us that they have further extended Maccoll's work. The results are as yet unpublished.

[3] Cf. Ref. 7, 8.

2. Fundamental equations for steady iso-energetic flows. For convenience of reference, we collect below the fundamental equations for steady iso-energetic flows. For flow in two-dimensions or with axial symmetry, a stream function may be defined, and a variational principle will be developed in terms of this stream function.

For steady three-dimensional motions, we have six hydrodynamical variables; the pressure p, the density ρ, the entropy S and the three components of velocity u_i $(i = 1, 2, 3)$. All these are to be determined as functions of the three spacial coordinates x_i $(i = 1, 2, 3)$ by the following six equations:

(i) the equations of motion (3),

$$u_j(\partial u_i/\partial x_j) = -\rho^{-1}(\partial p/\partial x_i), \tag{2.1}$$

(ii) the equation of continuity (1),

$$u_j(\partial \rho/\partial x_j) = -\rho(\partial u_j/\partial x_j) \tag{2.2}$$

(iii) the equation of state (1),

$$p = e^{S/c_v}\rho^{\gamma}, \tag{2.3}$$

where c_v is the specific heat at constant volume and γ is the ratio of specific heats; and finally

(iv) the equation of energy (1), which may be expressed in the form

$$u_j(\partial S/\partial x_j) = 0. \tag{2.4}$$

In addition, we often have the condition of iso-energetic flow

$$\gamma p/(\gamma - 1)\rho + q^2/2 = C, \tag{2.5}$$

where q is the magnitude of the velocity. The fact that this is a constant along each stream line may be derived from Eqs. (2.1) – (2.4). For flows derived from isentropic irrotational flows by a curved shock, C is actually a constant throughout the whole field. This is the case of isoenergetic flow under consideration. In general, (2.5) implies (2.4) when (2.1) and (2.2) hold. We note that (2.1) (2.2) and (2.5) form a system of five equations for p, ρ and u_i without reference to the entropy. One may therefore disregard the entropy at first and then supplement the results with the statement that the entropy, as given by (2.3), is a constant along each stream line, in accordance with (2.4).

For two-dimensional and axially symmetric flows, we have only five variables. We take the usual ϵ-convention and let (x, y) denote the distances along and perpendicular to the axis and u, v the velocity components in these directions. The four of the fundamental equations not involving entropy are:

$$u(\partial u/\partial x) + v(\partial u/\partial y) = (-1/\rho)(\partial p/\partial x) \tag{2.6}$$

$$u(\partial v/\partial x) + v(\partial v/\partial y) = (-1/\rho)(\partial p/\partial y) \tag{2.7}$$

$$\partial(\rho u)/\partial x + \partial(\rho v)/\partial y = -\epsilon \rho v/y, \tag{2.8}$$

where

$\epsilon = 0$ for two-dimensional motion, and

$\epsilon = 1$ for axial symmetry.

$$\gamma p/(\gamma - 1)\rho + q^2/2 = C. \tag{2.9}$$

The two equations involving entropy are the equation of state

$$p = e^{S/c_v}\rho^\gamma \tag{2.10}$$

and an integrated form of the equation of energy,

$$S = c_v f(\psi) \tag{2.11}$$

where ψ is the stream function defined by using (2.8):

$$\rho u y^\epsilon = \partial\psi/\partial y, \qquad \rho v y^\epsilon = -\partial\psi/\partial x \tag{2.12}$$

It is known (See, e.g., Vazsonyi [9], pp. 36, 37) that the vorticity is

$$\omega = (p/R)y^\epsilon(dS/d\psi). \tag{2.13}$$

In intrinsic form with θ denoting the direction of the stream line, and s, n denoting the arc lengths along and perpendicular to the stream line, the equations are

$$q\partial q/\partial s = (-1/\rho)(\partial p/\partial s), \tag{2.14}$$

$$\rho q^2(\partial\theta/\partial n) = -\partial p/\partial n, \tag{2.15}$$

$$\frac{1 - M^2}{q}\frac{\partial q}{\partial s} + \frac{\partial\theta}{\partial n} = -\epsilon\frac{\sin\theta}{y}, \tag{2.16}$$

$$\omega = \frac{\partial q}{\partial n} - q\frac{\partial\theta}{\partial s} = \frac{p}{\rho q R}\frac{\partial S}{\partial n}, \tag{2.17}$$

$$\partial\psi/\partial n = \rho q y^\epsilon. \tag{2.18}$$

In addition, equations (2.10) and (2.11) hold.

For irrotational flows,

$$\omega = 0, \tag{2.19}$$

and hence S is constant everywhere. Equation (2.10) may be written as

$$p = K\rho^\gamma \tag{2.20}$$

where K is a constant.

Variational principle. While variational principles [10] for irrotational isentropic flows have been well-known, it does not seem to have been pointed out that iso-energetic flows are also subject to a similar variational relation. It can be easily verified that

$$\delta I = \delta\int\int\left\{\left(C\rho - \frac{e^{f(\psi)}}{\gamma - 1}\rho^\gamma\right)y^\epsilon + \frac{1}{2\rho y^\epsilon}(\psi_x^2 + \psi_y^2)\right\}dx\,dy = 0, \tag{2.21}$$

with a preassigned function f and a fixed constant, C leads to the desired relations, provided $\rho(x, y)$ and $\psi(x, y)$ are allowed to vary independently, and proper boundary conditions are satisfied. As usual, the condition $\delta I = 0$ leads to

$$\iint\left\{\left(C - \frac{\gamma e^{f(\psi)}}{\gamma - 1}\rho^{\gamma-1}\right)y^s - \frac{1}{2\rho^2 y^s}(\psi_x^2 + \psi_y^2)\right\}\delta\rho\, dx\, dy$$

$$- \iint\left\{\frac{\rho^\gamma}{\gamma - 1}e^{f(\psi)}f'(\psi)y^s + \frac{\partial}{\partial x}\left(\frac{1}{\rho y^s}\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho y^s}\frac{\partial\psi}{\partial y}\right)\right\}\delta\psi\, dx\, dy$$

$$+ \int\frac{1}{y^s}\frac{1}{\rho}\frac{\partial\psi}{\partial n}\delta\psi\, dS = 0$$

Hence, if the boundary is a stream line with ψ preassigned, $\delta I = 0$ leads to

$$\frac{1}{y^{2s}}\frac{1}{2\rho^2}(\psi_x^2 + \psi_y^2) + \frac{\gamma e^{f(\psi)}\rho^{\gamma-1}}{\gamma - 1} = C,$$

and

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho y^s}\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho y^s}\frac{\partial\psi}{\partial y}\right) = -\frac{\rho^\gamma e^{f(\psi)}}{\gamma - 1}f'(\psi)y^s.$$

With p, u, v defined by (2.10) and (2.12), the first is the Bernoulli's equation (2.9), while the second equation is the vorticity equation (2.17). Thus, the fundamental equations are satisfied.

It may be noticed that the integrand in (2.21) (with $\epsilon = 0$) is again equal to the quantity $p + pq^2$ exactly as in the isentropic case.[4] Moreover, as formulated above, the integral can only be extended over a finite region. For an infinite region, a constant or a function independent of $\rho(x, y)$ and $\psi(x, y)$ has to be subtracted from the integrand to make the integral convergent.

3. Conditions across a shock. It is known that the pressure, density, and velocity behind a shock are obtainable from the corresponding quantities before the shock as long as the inclination of the shock to the incoming stream line is known. It will now be shown that if the curvature of the shock is known, the first spacial derivatives of the hydrodynamical quantities are also known. In general, if the flow in front of the shock is completely known, the spacial derivatives behind the shock are known to an order equal to that known for the derivatives of the angle of the shock.

One may regard this process as setting up the initial value problem after the shock. If the shock is given, the hydrodynamic quantities u, v, p, ρ, s are known behind it. The fundamental system of equations (2.6) – (2.11) can then be solved in terms of these initial conditions.

The calculations below will largely be carried out for the two-dimensional case; the results will be applied to discuss several problems in §4. The extension to the case with axial symmetry is almost immediate. Along the axis of symmetry,

[4] See Bateman, loc. cit., Eqs. (1) and (16).

however, the situation is more complicated. Thus, the extension of (4.1) and (4.2) to the conditions near a conical tip requires further investigation.

The pressure p, the density ρ, and the magnitude q and direction θ of the velocity in front and behind the shock satisfy the following relations (Fig. 1)

$$\rho_1 q_1 \sin \alpha = \rho_2 q_2 \sin (\alpha - \theta) \tag{3.1}$$

$$p_1 + \rho_1 q_1^2 \sin^2 \alpha = p_2 + \rho_2 q_2^2 \sin^2 (\alpha - \theta) \tag{3.2}$$

$$q_1 \cos \alpha = q_2 \cos (\alpha - \theta) \tag{3.3}$$

$$q_1 q_2 \sin \alpha \sin (\alpha - \theta) = c_*^2 - \frac{\gamma - 1}{\gamma + 1} q_1^2 \cos^2 \alpha \tag{3.4}$$

where

$$\theta = \theta_2 - \theta_1 , \qquad \alpha = \alpha_1 - \theta_1 , \qquad \tfrac{1}{2}[(\gamma + 1)/(\gamma - 1)]c_*^2 = C. \tag{3.5}$$

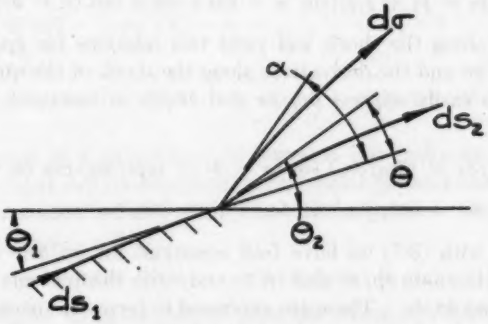

FIG. 1. CONDITIONS AT A CURVED SHOCK

In general, a quantity in front of the shock will carry the subscript 1; behind a shock, the subscript 2. This convention will be used throughout the paper. From (3.1) – (3.5) it can be shown that the jump in entropy is

$$S_2 - S_1 = c_v \log \left(\frac{2\gamma}{\gamma + 1} M_1^2 \sin^2 \alpha - \frac{\gamma - 1}{\gamma + 1} \right) \cdot \left(\frac{\gamma - 1}{\gamma + 1} \sin^2 \alpha + \frac{2}{(\gamma + 1) M_1^2} \right)^{\gamma} . \tag{3.6}$$

There are *ten* spacial derivatives, two for each of the *five* quantities p, ρ, q, θ, and S. By (2.10), the calculation of the derivatives of S may be expressed in terms of those for p and ρ. More explicitly, the derivatives of S are $\partial S/\partial s = 0$, $\partial S/\partial n = \rho q dS/d\psi$ with $dS/d\psi$ calculated from (3.6). Thus, we need to consider only the eight derivatives of p, ρ, q, θ. Equations(3.1)–(3.4) will furnish the four derivatives along the shock. These furnish four relations for the tangential and normal derivatives defined with respect to a stream line. The other four are supplied by (2.9), and (2.14)–(2.16).

Furthermore, if we eliminate $\partial q/\partial s$ between (2.14) and (2.16) we have the relations

$$\begin{cases} \rho q^2(\partial\theta/\partial n) = (1 - M^2)(\partial p/\partial s), \\ \rho q^2(\partial\theta/\partial s) = -\partial p/\partial n, \end{cases} \tag{3.7}$$

which are two relations among the four spatial partial derivatives of p and θ. If we now eliminate ρ_2 and q_2 from Eqs. (3.1) to (3.4), we have only to deal with the two quantities p and θ. After the derivatives of p and θ are found, one can easily calculate those for q from (1.14) and (1.17) with $dS/d\psi$ from (3.6) and those for ρ from (2.9). Let us now carry out these calculations.

Elimination of ρ_2 and q_2 from (3.1) and (3.4) gives

$$\begin{cases} p_2 = p_1 + \rho_1\left\{q_1^2\left(1 - \dfrac{2}{\gamma+1}\cos^2\alpha\right) - c_*^2\right\}, \\ p_2 = p_1 + \rho_1 q_1^2\{\sin^2\alpha - \sin\alpha\cos\alpha\tan(\alpha - \theta)\}, \end{cases} \tag{3.8}$$

Differentiation along the shock will yield two relations for $\partial p_2/\partial\sigma$ and $\partial\theta_2/\partial\sigma$ in terms of $\partial\alpha_1/\partial\sigma$ and the derivatives along the shock of the quantities in front of it. One can easily express $\partial p_2/\partial\sigma$ and $\partial\theta_2/\partial\sigma$ in terms of $\partial p_2/\partial s$, $\partial p_2/\partial n$, $\partial\theta_2/\partial S$, $\partial\theta_2/\partial n$:

$$\begin{cases} \partial p_2/\partial\sigma = (\partial p_2/\partial n_2)\sin(\alpha - \theta) + (\partial p_2/\partial s_2)\cos(\alpha - \theta), \\ \partial\theta_2/\partial\sigma = (\partial\theta_2/\partial n_2)\sin(\alpha - \theta) + (\partial\theta_2/\partial s_2)\cos(\alpha - \theta). \end{cases} \tag{3.9}$$

Thus, together with (3.7) we have four equations for $\partial p/\partial s_1 \cdots, \partial\theta/\partial n$. Actually we shall eliminate $\partial p/\partial n$ and $\partial\theta/\partial n$ and write the final relations in terms of two for $\partial p/\partial s$ and $\partial\theta/\partial s$. These are expressed in terms of spacial derivatives of the quantities on the left-hand side of the shock. Again, in front of the shock we may eliminate all other derivatives in favor of $\partial p/\partial s$ and $\partial\theta/\partial s$. The calculation is even simpler here because the motion is irrotational. It can be easily verified that

$$\begin{cases} \partial p_1/\partial\sigma = (\rho_1 q_1^2/r_1)\sin\alpha + (\partial p_1/\partial s_1)\cos\alpha, \\ \partial\rho_1/\partial\sigma = (\rho_1 M_1^2/r_1)\sin\alpha + (M_1^2/q_1^2)(\partial p_1/\partial s_1)\cos\alpha, \\ \partial q_1/\partial\sigma = -(q_1/\gamma_1)\sin\alpha - (1/\rho_1 q_1)(\partial p_1/\partial s_1)\cos\alpha, \\ \partial\theta_1/\partial\sigma = -(M_1^2 - 1)(\rho_1 q_1^2)^{-1}(\partial p_1/\partial s_1)\sin\alpha - (1/r_1)\cos\alpha. \end{cases} \tag{3.10}$$

The final equations obtained are

$$\begin{aligned} A_1(\rho_1 q_1^2)^{-1}(-\partial p_1/\partial s_1) + B_1(\partial\theta_1/\partial s_1) \\ = A_2(\rho_1 q_1^2)^{-1}(-\partial p_2/\partial s_2) + B_2(\partial\theta_2/\partial s_2) + C(\partial\alpha_1/\partial\sigma), \end{aligned} \tag{3.11}$$

$$\begin{aligned} A_1'(\rho_1 q_1^2)^{-1}(-\partial p_1/\partial s_1) + B_1'(\partial\theta_1/\partial s_1) \\ = A_2'(\rho_1 q_1^2)^{-1}(-\partial p_2/\partial_2 s) + B_2'(\partial\theta_2/\partial s_2) + C'(\partial\alpha_1/\partial\sigma), \end{aligned} \tag{3.12}$$

where the coefficients are functions of M_1, α, and θ given in Table I.

TABLE 1

	α, θ general	$\alpha = \pi/2, \theta = 0$
A_1	$\frac{2}{\gamma+1}\cos\alpha\left[(3M_1^2-4)\sin^2\alpha+\frac{1-\gamma}{2}\right]$	0
B_1	$\frac{2}{\gamma+1}\sin\alpha\left[(M_1^2-4)\sin^2\alpha+\frac{5-\gamma}{2}\right]$	$\frac{2}{\gamma+1}\left[M_1^2-\frac{\gamma+3}{2}\right]$
A_2	$\cos(\alpha-\theta)$	0
B_2	$\sin 2\alpha/2\cos(\alpha-\theta)$	q_2/q_1
C	$\frac{2}{\gamma+1}\sin 2\alpha$	0
A_1'	$M_1^2\cos^2\alpha\cos\theta-(M_1^2-1)\cos(2\alpha+\theta)$	M_1^2-1
B_1'	$M_1^2\sin^2\alpha\sin\theta+\sin(2\alpha+\theta)$	0
A_2'	$1+(M_2^2-2)\sin 2(\alpha-\theta)$	M_2^2-1
B_2'	$\sin 2\alpha$	0
C'	$\sin 2\theta/2\cos(\alpha-\theta)$	$(q_2/q_1)-1$

The predominance of p and θ may be regarded physically as follows. The isentropic flow in front can be described by one scalar (p) and one direction (θ). The shock relations allow all the quantities to be expressed in terms of them, and the shock angle α_1. Thus,

$$p_2 = p_2(p_1, \theta_1, \alpha_1), \qquad \theta_2 = \theta_2(p_1, \theta_1, \alpha_1)$$

The pressure gradient along the shock thus introduces the spatial derivatives which may be handled by (3.7) and (3.9). Notice that (3.7) follows directly from the intrinsic forms (2.14)–(2.16) of the equations of motion and continuity and holds either behind or in front of the shock.

4. Shocks attached to the solid boundary. In this section, we shall apply our results to a few special cases of interest. The first is the case of a uniform supersonic flow past a body with a high enough Mach number. The shock is then attached to the nose of the body. This case has been considered by Crocco [2] and Thomas [4]. Next we shall consider the shock attached to a body with continuous directions but possibly with discontinuities in curvature. In this case θ approaches zero. The weak family of the shock is thus expected to become a Mach line (characteristic), while the strong family is expected to become a normal shock. The latter case has been considered by Hantsche [3], Emmons [7] and Tsien [8]. The results obtained here are somewhat different from theirs. Here it is concluded that a shock can exist in the case of continuous convex curvature only for Mach number above a certain critical value. This might be understood to mean that the interaction between shock and boundary layer would be different for Mach numbers above or below this critical value. Detailed discussions, including physical interpretations, will be given below.

(a) Shock attached to the sharp nose of a body in uniform supersonic stream. The simplest problem to which the above considerations may be applied is the shock formed by a uniform supersonic flow past a symmetrical body at a sufficiently high Mach number. In this case, all the quantities bearing the subscript 1 are constants, and the equations (3.11) and (3.12) become

$$\frac{1}{r_2}\sin 2\alpha\left\{\frac{2}{\gamma+1}\sin 2\alpha - \frac{\sin 2\theta}{4\cos^2(\alpha-\theta)}\right\} = -\frac{1}{\rho_1 q_1^2}\frac{\partial p_2}{\partial s_2}\left\{\frac{2}{\gamma+1}\sin 2\alpha[1+\sin^2(\alpha-\theta)(M_1^2-2)] - \frac{\sin 2\theta}{2}\right\}, \tag{4.1}$$

$$\frac{1}{r_2}\sin 2\alpha\left\{\cos(\alpha-\theta) - \frac{1+(M_1^2-2)\sin^2(\alpha-\theta)}{2\cos(\alpha-\theta)}\right\} = -\frac{\partial \alpha_1}{\partial \sigma}\left\{\frac{2}{\gamma+1}\sin 2\alpha[1+(M_1^2-2)\sin^2(\alpha-\theta)] - \frac{\sin 2\theta}{2}\right\}. \tag{4.2}$$

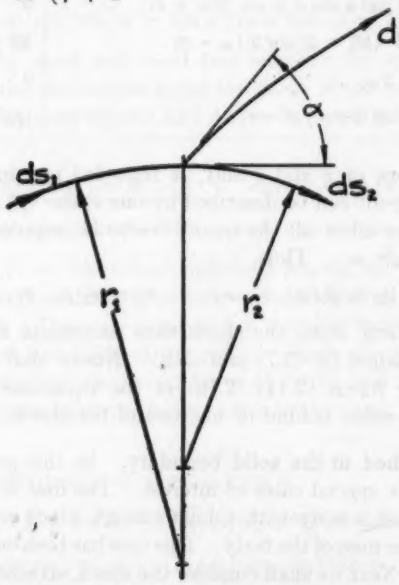

FIG. 2. CONDITIONS AT A CURVED MACH LINE

These equations were obtained by Crocco and Thomas. It is interesting to note that the curvature of the shock and the pressure gradient behind it are both proportional to the curvature of the body, with coefficients of proportionality depending on the free stream Mach number and the angle of the body.

(b) Shocks attached to boundary with continuous slope—weak family. In the limit $\theta \to 0$, the discontinuity becomes one of higher order. The first three shock conditions (3.1)–(3.4) are satisfied by

$$p_1 = p_2, \qquad \rho_1 = \rho_2, \qquad q_1 = q_2, \tag{4.3}$$

while the last one leads to the conclusion that α becomes equal to the Mach angle $\sin^{-1}(1/M_1)$. The first order relations (3.11) and (3.12) lead to

$$\frac{\partial p_2}{\partial s_2} - \frac{\partial p_1}{\partial s_1} = \rho_1 q_1^2 \left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tan \alpha \tag{4.4}$$

$$\frac{\partial \alpha_1}{\partial \sigma} = \frac{4 + (\gamma - 3)M_1^2}{2\sqrt{M_1^2 - 1}} \left(\frac{\sin \alpha}{r_1} + \frac{1}{\rho_1 q_1^2} \frac{\partial p_1}{\partial s_1} \cos \alpha\right). \tag{4.5}$$

The curvature of the Mach line is conditioned by two factors, the value of the pressure gradient along the Mach line and the multiplicative coefficient. Note that the latter quantity changes sign when $M_1^2 = 4/(3 - \gamma)$(for air, $\gamma = 1.4$ and $M_1 = \sqrt{2.5}$), and that the curvature may be positive or negative. Note also that the curvature of the Mach line is independent of the curvature $1/r_2$ of the stream line. If $r_1 = r_2$, there would not be any discontinuity in the first derivatives of physical quantities.

The relation (4.4) can be obtained more directly from the condition for characteristics. It is known that the discontinuity in derivatives of velocity must propagate along the characteristic, which is at an angle $\alpha = \sin^{-1}(1/M)$ with the direction of the stream line (Fig. 2). The pressure is continuous across it. Hence, if the jump is denoted by [ ], $[\partial p/\partial \sigma] = 0$. On the other hand,

$$\partial p/\partial \sigma = (\partial p/\partial s) \cos \alpha + (\partial p/\partial n) \sin \alpha,$$

and hence

$$0 = [\partial p/\partial s] \cos \alpha + [\partial p/\partial n] \sin \alpha$$

which then yields (4.4), if $\left[\frac{\partial p}{\partial n}\right]$ is evaluated by (2.15).

(c) Shocks attached to boundary with continuous direction—strong family. The limit of the strong family is given by $\theta \to 0$, $\alpha \to \pi/2$. In this case, the coefficients A_1, B_1, A_2, B_2, C and A_1', B_1', A_2', B_2', C′ tend to the values shown in the last column of Table I. If all the other quantities in equations (3.11) and (3.12) remain finite, those equations become

$$\frac{2}{\gamma + 1}\left\{M_1^2 - \frac{\gamma + 3}{2}\right\}\left(-\frac{1}{r_1}\right) - \frac{1}{\gamma + 1}\left\{\frac{2}{M_1^2} + (\gamma - 1)\right\}\left(-\frac{1}{r_2}\right) = 0 \tag{4.6}$$

This is the relation discussed by previous authors.[5] However, if the curvature of the shock is allowed to become infinite, then it turns out that

$$\frac{2}{\gamma + 1}\left\{M_1^2 - \frac{\gamma + 3}{2}\right\}\left(-\frac{1}{r_1}\right) - \frac{1}{\gamma + 1}\left\{\frac{2}{M_1^2} + (\gamma - 1)\right\}\left(-\frac{1}{r_2}\right) \leq 0. \tag{4.7}$$

[5] See, e.g. Tsien, loc. cit., p. 72, Eq. (16).

The detailed calculations leading to this relation are given below. The implications of this relation are very interesting. They will be discussed in detail in the next section. Physical interpretations of the results will also be indicated there.

Calculation of the limit form of (3.11) *and* (3.12).

In carrying out the limiting process, we shall take the quantities in front of the shock to be finite, as the shock can only influence flow conditions downstream. Thus, the only quantities which may become infinite are

(i) the curvature of the shock, $\partial\alpha_1/\partial\sigma$

and

(ii) the pressure gradient behind the shock, $-(\rho_1 q_1^2)^{-1}(\partial p_2/\partial s_2)$.

The limit of (3.11) becomes

$$\frac{2}{\gamma+1}\left\{M_1^2-\frac{\gamma+3}{2}\right\}\left(-\frac{1}{r_1}\right)=\lim A_2\left(-\frac{1}{\rho_1 q_1^2}\frac{\partial p_2}{\partial s_2}\right)+\frac{q_2}{q_1}\left(-\frac{1}{r_2}\right)+\lim C\left(\frac{\partial\alpha_1}{\partial\sigma}\right),$$

or

$$\frac{2}{\gamma+1}\left(M_1^2-\frac{\gamma+3}{2}\right)\left(-\frac{1}{r_1}\right)-\frac{1}{\gamma+1}\left(\frac{2}{M_1^2}+\gamma-1\right)\left(-\frac{1}{r_2}\right)=\lim A_2\left(-\frac{1}{\rho_1 q_1^2}\frac{\partial p_2}{\partial s_2}\right)+\lim C\left(\frac{\partial\alpha_1}{\partial\sigma}\right). \tag{4.8}$$

The limit of (3.12) does not lead to a finite result. However, if we multiply the equation by A_2 before taking the limit, there is obtained

$$0=A_2'\lim A_2\left(-\frac{1}{\rho q_1^2}\frac{\partial p_2}{\partial s_2}\right)+C'\lim A_2\left(\frac{\partial\alpha_1}{\partial\sigma}\right). \tag{4.9}$$

If we eliminate the term in pressure gradient in (4.8) by means of (4.9), we obtain

$$\frac{2}{\gamma+1}\left\{M_1^2-\frac{\gamma+3}{2}\right\}\left(-\frac{1}{r_1}\right)-\frac{1}{\gamma+1}\left\{\frac{2}{M_1^2}+\gamma-1\right\}\left(-\frac{1}{r_2}\right)=\left\{1-\frac{C'}{A_2'}\lim\frac{A_2}{C}\right\}\lim C\left(\frac{\partial\alpha_1}{\partial\sigma}\right). \tag{4.10}$$

Now, by (3.5),

$$\lim C\left(\frac{\partial\alpha_1}{\partial\sigma}\right)=\lim\frac{2}{\gamma+1}\sin 2\alpha\left(\frac{\partial\alpha}{\partial\sigma}-\frac{\partial\theta_1}{\partial\sigma}\right)=-\lim\frac{1}{\gamma+1}\frac{\partial}{\partial\sigma}\cos 2\alpha.$$

Hence,

$$\lim C(\partial\alpha_1/\partial\sigma)\leq 0, \tag{4.11}$$

since $\cos 2\alpha$ is least for $\alpha = \pi/2$. By using the explicit expressions for A_2, C and A_2', C' and the relation

$$\lim_{\substack{\alpha \to \pi/2 \\ \theta \to 0}} \frac{\sin 2\alpha}{2 \cos (\alpha - \theta)} = \frac{q_2}{q_1},$$

(See the values of B_2 in Table I), it can be easily verified that

$$1 - \frac{C'}{A_2'} \lim \frac{A_2}{C} = 1 + \left(\frac{q_1}{q_2} - 1\right) \frac{1}{1 - M_2^2} \frac{1}{\gamma + 1} > 0. \tag{4.12}$$

By putting (4.11) and (4.12) into (4.10), the relation (4.7) is verified. It should further be noted that a finite value for (4.10) may be obtained only when

$$\lim \partial(\cos 2\alpha)/\partial\sigma = \text{finite} \tag{4.13}$$

This point will be discussed further in the next section.

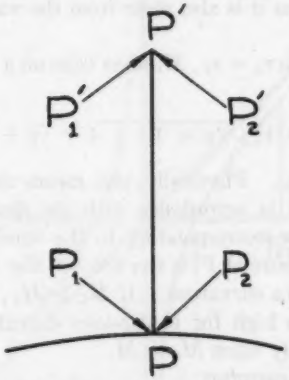

FIG. 3. PRESSURE RELATIONS AT A STRAIGHT NORMAL SHOCK

5. Further studies of normal shocks over solid boundaries. In this section, we shall discuss further the results obtained in §3(c). The implications and physical interpretations will be given. First, we shall consider the case of the finite curvature; next, the modification of the situation due to infinite curvature will be discussed. The shape of a shock with infinite curvature will then be discussed.

(i) *Shocks with finite curvature.*

In the case of finite curvature, the equality sign holds in (4.7). We then have

$$r_2 = r_1 \left(\frac{1}{M_1^2} + \frac{\gamma - 1}{2}\right) \Big/ \left(M_1^2 - \frac{\gamma + 3}{2}\right). \tag{5.1}$$

This is the result obtained by previous authors. It shows that for a given Mach number, the *ratio* of the curvatures[6] must have a certain value, in order that a

[6] Note that in the case of a straight boundary, regarded as a limiting case of curved

normal shock may be formed. We propose to give here the physical interpretation of this result, which does not seem to have been fully discussed.

For definiteness, let us first consider a straight shock intersecting the solid boundary at P, and let us compare the conditions at P with that at a neighboring point P' (Fig. 3). If the conditions at P_1 are known, the normal pressure gradient is also known. Thus, we have a knowledge of the flow conditions at P_1'. Then, from the shock conditions, the flow conditions at P_2 and P_2' are also known. In particular, the pressure gradient along P_2P_2' is known. As this must balance the centrifugal force, the curvature of the solid boundary cannot be arbitrary. This explains why r_2 is completely determined by r_1 in the case of a straight shock.

If the shock has a finite curvature, then the shock angle at P' differs from $\pi/2$ by a quantity of the order of PP', a small quantity of the first order. However, the change of pressure jump across the shock at P' is second order in the change of the angle, as we are in the neighborhood of the normal shock. A reference to the shock polar shows this clearly. Thus a finite curvature of the shock does not modify the situation, as it is also clear from the way eq. (3.11) simplifies to (4.6).

Eq. (5.1) shows that with $r_1 = r_2$, M_1 must take on a critical value M_c defined by

$$M_c^2 = \tfrac{1}{2}\{\sqrt{(\gamma+1)^2+4} + (\gamma+1)\}. \tag{5.2}$$

If $M_1 < M_c$, then $r_2 > r_1$. Physically, this means that the pressure gradient developed after the shock in accordance with the shock conditions is able to balance a centrifugal force corresponding to the smaller curvature $1/r_2$ (r_1, r_2 both positive[7]). The pressure at P_2' is too low for the shock to be formed over a boundary with continuous curvature. If $M_1 > M_c$, the opposite is true; i.e., the pressure at P_2' is too high for continuous curvature of boundary. The pressure is just suitable only when $M_1 = M_c$.

(ii) *Shocks with infinite curvature.*

If the shock has an infinite curvature, the difference of shock angle between P and P' is no longer of the same order as PP'. Therefore, the normal pressure gradient after the shock may be changed by finite values. This is borne out by the fact that the limiting values of $C(\partial\alpha_1/\partial\sigma)$ in (4.8) may not vanish, if infinite curvature is allowed. However, the adjustment appears to be one-sided as the inequality sign in (4.7) indicates. Indeed, in the case of continuous curvature ($r_1 = r_2$), it can be easily deduced from (4.7) that (cf. Fig. 4)

$$\begin{aligned} M_1^2 - \frac{1}{M_1^2} &> \gamma + 1 \quad \text{if} \quad r_1 = r_2 > 0, \\ M_1^2 - \frac{1}{M_1^2} &< \gamma + 1 \quad \text{if} \quad r_1 = r_2 < 0 \end{aligned} \tag{5.3}$$

boundary with $1/r_1$, $1/r_2$ approaching zero, the limit may take on any value. Hence, the shock may occur with any Mach number.

[7] This is taken to fix our ideas. The statement can be easily modified to suit other cases.

For the equality sign to hold, $M_1 = M_c$. Thus, to satisfy these requirements, we must have

$$\begin{cases} M_1 > M_c & \text{if} \quad r_1 = r_2 > 0 \\ M_1 < M_{c_1} & \text{if} \quad r_1 = r_2 < 0. \end{cases} \tag{5.4}$$

That is, *over a convex (concave) surface, a normal shock can be formed only when the Mach number of the flow is above (below) a certain critical Mach number.* This conclusion is interesting enough to require a further clarification of the physical situation.

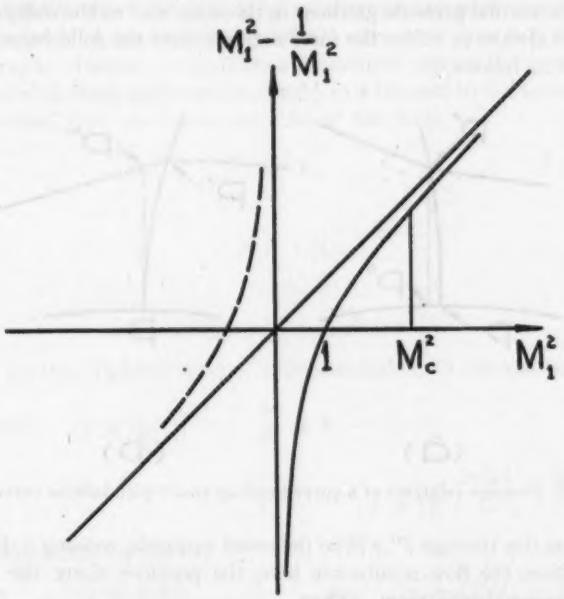

FIG. 4. DIAGRAM SHOWING FUNCTION DETERMINING THE CRITICAL MACH NUMBER

To fix our ideas, let us restrict ourselves to the case of convex surface ($r_1 = r_2 > 0$). It has been pointed out above that the pressure at P_2' is too high for continuous curvature of the boundary if $M_1 > M_c$. Thus, if the shock at P' is oblique, the pressure rise across the shock would be reduced, and the pressure gradient after the shock would balance the centrifugal acceleration at P_2 after the shock. If $M_1 < M_c$, the pressure at P_2' is too low even with a normal shock. Thus, a finite shock angle at P' would cause the pressure after it even more unbalanced.

However, this is not the whole story. We are considering the pressure gradient along the shock, and not the normal pressure gradient. Because of the infinite curvature of the shock, these two could be different. The geometry and the

pressure distribution after the shock must be examined more carefully. This is also borne out by the fact that there are two types of terms in the limiting process of (4.8): one is related to the infinite shock curvature, the other is related to the infinite pressure gradient. The former has been discussed above, the latter has just been brought up.

Let us fix our attention on Fig. 5(a) first. We notice that the normal pressure gradient must be calculated on the basis of the pressure difference $p(P_2') - p(P'')$ and not for $p(P_2') - p(P_2)$. The difference between these two might be significant if the pressure gradient along the solid boundary is infinite; and this is indeed the case. It is interesting to note that this infinite pressure gradient influences the normal pressure gradient in the same way as the obliqueness of the shock. It is always to *reduce* the force acting *toward* the solid boundary. This may be seen as follows.

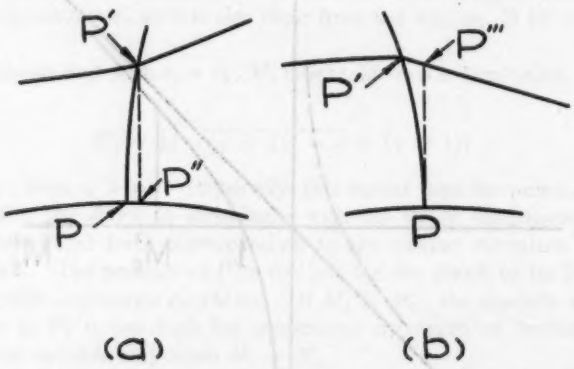

FIG. 5. Pressure relations at a curved normal shock with infinite curvature.

The stream line through P' will be deflected upwards, causing a divergence of the flow. Since the flow is subsonic here, the pressure along the stream line will then increase downstream. Thus,

$$p(P'') > p(P_2)$$

Hence,

$$-\{p(P'') - p(P_2')\} - \{p(P_2') - p(P_2)\} = p(P_2) - p(P'') < 0$$

In the limiting process of taking the pressure gradient, although the distance P_2P'' goes to zero as a higher order of P_2P_2', the infinite pressure gradient would cause the difference $p(P'') - p(P_2)$ to make a finite contribution. This explains the two types of terms considered in (4.8). However, because of the inequality sign in the above formula, it is clear that the normal pressure gradient can only be *smaller* than that along the shock. Thus, the infinite pressure gradient along the boundary acts in the same way as the obliqueness of the shock.

It might be thought that if the shock were curved in the other direction (Fig.

5(b)), the conclusion would be different. However, in this case, the flow is convergent after the shock; hence,

$$p(P''') < p(P_2')$$

Thus,

$$p(P''') - p(P_2) < p(P_2') - p(P_2),$$

and the normal pressure gradient is again smaller than the pressure gradient along the shock.

Having clarified the physical interpretation of the one-sided nature of the influence of infinite shock curvature, let us examine the geometry of the shock somewhat further. A curve with an infinite curvature does not show corners in a usual graph. Indeed, if one takes a coordinate system (ξ, η) with origin at the intersection of shock and boundary and ξ-axis tangent to the boundary, it can be easily verified that the curve must be of the form

$$\xi = k\eta^{3/2} \qquad (k = \text{constant}) \tag{5.5}$$

to make

$$\alpha = \pi/2$$

and

$$d \cos 2\alpha/d\sigma = \text{constant}$$

at $(\xi, \eta) = (0, 0)$. Indeed, after a little calculation, (4.10) can be reduced to

$$\{2M_1^2 - (\gamma + 3)\} \frac{1}{r_1} + \left\{ \frac{2}{M_1^2} + (\gamma - 1) \right\} \frac{1}{r_2} = \frac{9k^2}{2} \left\{ 1 + \frac{1}{(\gamma + 1)^2} \frac{2\gamma M_1^2 - (\gamma - 1)}{1 + (\gamma - 1)M_1^2/2} \right\}. \tag{5.6}$$

In this reduction, the formulae

$$\frac{q_1}{q_2} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2}, \qquad 1 - M_2^2 = \frac{(\gamma + 1)(M_1^2 - 1)}{2\gamma M_1^2 - (\gamma - 1)}$$

for normal shocks are used.

6. Flow behind detached shocks. So far we have been dealing with shocks attached to solid bodies. When a sharp-nosed body is placed in a supersonic stream, this is the case, when the Mach number of the flow is very high. In fact, the flow behind the shock is then often supersonic, and can be worked out by stepwise numerical integration with the help of characteristics. Equations (3.1) and (3.2) would then be very useful in getting better accuracy. However, for axially symmetrical flows, the corresponding formulae have not yet been worked out.

If the Mach number of the flow is relatively low, or if the body is blunt, the

shock detaches from the body, and there is always a subsonic region behind the shock. Maccoll and others have considered the cases of a supersonic flow past a body with straight boundaries followed by a sharp corner at the shoulder. In this way, he is able to define the subsonic region to a certain extent. A numerical procedure is then developed to study this region, treating the problem as a boundary value problem.

This method does not apply to bodies with blunt noses and smooth boundaries. Also in common with all numerical methods, it is difficult to obtain general conclusions. A number of specific cases have to be computed in order that general conclusions may be drawn.

On the other hand, experiments have been carried out for supersonic flow past spheres (cf. Fig. 6). Also, if a hot wire is ever used in a supersonic stream, such a fine hot wire is presumably in the form of a circular cylinder. It therefore seems desirable that methods be developed to investigate such flows.

The investigations below are aimed at such problems. The method of approach can be best understood by considering the limiting case of very high Mach numbers. It is known that the shock would be then very close to the body. Considering the local situation at the nose of the body, it is clear that the most important characteristic length is the radius of curvature of the body. Thus, it might be expected that the radius of curvature of the shock and the distance of the shock from the body are both proportional to the radius of curvature of the body, with coefficients depending on the Mach number of the uniform stream. Hence, if one would investigate the local nature of the flow in front of the nose of the body and that behind the "'nose" of the shock, and then try to fit the two flows together, these ratios can be found.

To improve on this first-order theory, one may develop the power series expansions in front of the nose of the body and behind the nose of the shock. It will be shown below that definite results are obtainable to any order of approximation desired; i.e., the power series may be terminated at any power, still the shape of the shock may be determined in terms of the shape of the body, and the flow in between would also be known, including the distance of the shock from the body.

The validity of the method depends on two conditions. First, the power series developed in the neighborhood of the nose of the body may be applied up to and including the neighborhood of the nose of the shock, and vice versa. Secondly, these power series may include part of the supersonic region. If this be the case, it is clear that any modification of the shape of the body downstream from the region of interest to us will not disturb the flow conditions in the subsonic region (cf. Fig. 6).

These conditions cannot be verified a priori. However, in the case of the sphere, the results obtained along these lines seem to be in reasonable agreement with experiments. In the case of bodies of extreme shape, such as a finite cylinder with one end placed against the stream, it is obvious that the method would break down. In the extreme case of very high Mach numbers, the argument put forth about a local characteristic length obviously does not hold.

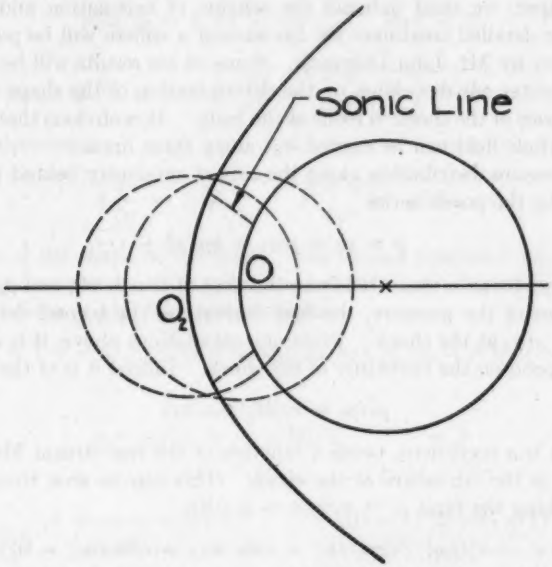

FIG. 6. Detached shock in front of a symmetrical body placed in a uniform stream.

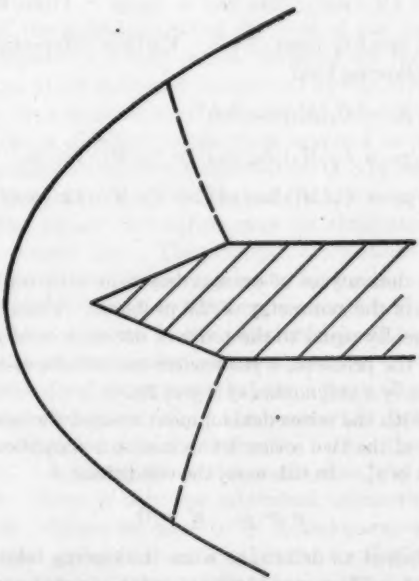

FIG. 7. Detached shock showing the sonic line through the sharp shoulder of the body.

In this paper, we shall indicate the scheme of calculation and state some results. The detailed treatment for the case of a sphere will be published in a separate paper by Mr. John Dugundji. Some of his results will be stated here.

We shall center our discussion on the determination of the shape of the shock and the distance of the shock in front of the body. It is obvious that the calculation of the whole field can be carried out along these lines.

Let the pressure distribution along the axis of symmetry behind the shock be represented by the power series

$$p = p_2 + p_2' x_2 + \tfrac{1}{2} p_2'' x_2^2 + \cdots \tag{6.1}$$

where x_2 is the distance measured from the nose of the shock, and $p_2, p_2', p_2'', \cdots$ are the values of the pressure, the first derivative, the second derivative, etc. with respect to x_2 at the shock. From our calculations above, it is obvious that p_2' should depend on the curvature of the shock. Indeed it is of the form

$$p_2'/p_0 = c_1(M_1)(\partial\alpha/\partial\sigma) \tag{6.2}$$

where $c_1(M_1)$ is a coefficient, being a function of the free-stream Mach number, and $(\partial\alpha/\partial\sigma)$ is the curvature of the shock. This can be seen from (3.11) and (3.12) by taking the limit $\alpha = \pi/2$, $\theta = 0$ with

$$-(\rho_1 q_1^2)^{-1}(\partial p_1/\partial s_1) = (\partial\theta_1/\partial s_1) = (\partial\theta_2/\partial s_2) = 0.$$

We have

$$(M_2^2 - 1)(-\rho_1 q_1^2)^{-1}(\partial p_2/\partial s_2) + (q_2/q_1 - 1)(\partial\alpha/\partial\sigma) = 0,$$

from (3.12), and an identity from (3.11). Further differentiation will then lead to relations of the following kind

$$\begin{cases} p_2''/p_0 = C_2(M_1)(\partial\alpha/\partial\sigma)^2 \\ p_2'''/p_0 = C_3(M_1)(\partial\alpha/\partial\sigma)^3 + C_4(M_1)(\partial^3\alpha/\partial\sigma^3) \\ p_2^{iv}/p_0 = C_5(M_1)(\partial\alpha/\partial\sigma)^4 + C_6(M_1)(\partial\alpha/\partial\sigma)(\partial^3\alpha/\partial\sigma^3) \\ \cdots \end{cases} \tag{6.3}$$

We notice that the derivatives of even orders of α with respect to σ all vanish identically, because of the symmetry of the problem. Thus, the order of derivatives introduced is really equal to the order of derivatives of p_2. Thus, because of the symmetry of the problem, *n parameters are introduced in order that we may represent the pressure by a polynomial of degree $2n$.*

Before going on with the series development around the nose of the body, and the fitting together of the two series, let us notice a simplification if we stop the series (6.1) at terms in x_2^4. In this case, the conditions

$$p = p_0, \quad p' = 0$$

at the nose are sufficient to determine some interesting relations. Let the distance in between be δ. Then, we obtain two relations of the type

$$p(\delta)/p_0 = F\left(\delta\frac{\partial\alpha}{\partial\sigma}, \delta^3\frac{\partial^3\alpha}{\partial\sigma^3}, M_1\right) = 1,$$

$$p'(\delta)/p_0 = F\left(\delta\frac{\partial\alpha}{\partial\sigma}, \delta^3\frac{\partial^3\alpha}{\partial\sigma^3}, M_1\right) = 0.$$

These then determine

$$\delta\frac{\partial\alpha}{\partial\sigma} = f(M_1), \qquad \delta^3\frac{\partial^3\alpha}{\partial\sigma^3} = g(M_1) \tag{6.4}$$

independent of the shape of the body. The second relation is presumably not accurate, but one may expect the first one to hold reasonably well.

The calculations of Mr. John Dugundji give

$$p_2/p_0 = 1 + \pi(M_1),\ p_2'/p_0 = \mu(M_1)(\partial\alpha/\partial\sigma),\ p_2''/p_0 = \nu(M_1)(\partial\alpha/\partial\sigma)^2, \tag{6.5}$$

with

$$\pi = \{1 + (\gamma-1)M_2^2/2\}^{-\gamma/(\gamma-1)} - 1$$

$$\mu = -4\gamma(\gamma+1)^{-1}\{1 + (\gamma-1)M_2^2/2\}^{-\gamma/(\gamma-1)} \tag{6.6}$$

$$\nu = \frac{8\gamma}{(1-M_2^2)^2}\,\frac{\{1+(\gamma-1)M_2^2/2\}^{-\gamma/(\gamma-1)}}{2\gamma M_1^2 - (\gamma-1)}\left\{\frac{(M_1^2-1)^2[2(\gamma M_2^4+1)+(1-M_2^2)^2]}{(M_2^2-1)[2+(\gamma-1)M_1^2]} + (M_1^2-1) + M_1^2(1-M_2^2)\right\}.$$

The expansion in the neighborhood of the nose of the body is more complicated, because it depends not only on the shape of the body, but also on the entropy distribution, which in turn is determined by the curvature of the shock. However, it seems clear from detailed calculations (cf. following section), that the $(2n)$th derivative of pressure at the nose involves at most $\partial^n\alpha/\partial\sigma^n$. The other additional parameters are the n derivatives of odd orders of the angle of the boundary with respect to its arc length, and the first derivative of the velocity along the axis. The higher derivatives may be eliminated by requiring the boundary to be a stream line. The detailed calculations have been carried through including third powers in the expression for pressure (see eqs. (7.27) and (7.28)). It is seen that up to this order of approximation, the change of entropy has not yet entered the problem in the two-dimensional case, but has already influenced the results in the axially symmetrical case.

By fitting the series developments around the nose of the shock and the nose of the body, one can determine all of the parameters entering the problem. Suppose one uses both series up to the degree $2n$. Then there are n parameters from the series (6.1). Applying this series at the nose, one introduces an additional parameter δ. There is only one additional unknown introduced by the series near the body. Thus, we have $(n+2)$ unknowns all together. If we differentiate the power series (6.1) $(n+1)$ times to obtain the value of $p(\delta)$, $p'(\delta), \cdots p^{(n+1)}(\delta)$ and equate them to the value at the nose, we can obtain all

these parameters. The worst approximation in this calculation is presumably in $p^{(n+1)}(\delta)$. But even here, we have a polynomial of degree $n-1$, and the accuracy can be improved by increasing n.

In practice, it is probably better to use derivatives of lower orders, and compare their values at both noses by differentiating both series. Thus, the accuracy can be improved.

From dimensional considerations, it is obvious that the results should be expressible in the form

$$\begin{cases} \delta \dfrac{\partial \alpha}{\partial \sigma} = f_1 \left\{ \delta \dfrac{\partial \alpha_0}{\partial \sigma_0}, \delta^3 \dfrac{\partial^3 \alpha_0}{\partial \sigma_0^3}, \cdots, \delta^{2n-1} \dfrac{\partial^{2n-1} \alpha_0}{\partial \sigma_0^{2n-1}}, M_1 \right\}, \\ \cdots \\ \delta \dfrac{\partial^{2n-1} \alpha}{\partial \sigma^{2n-1}} = f_n \left\{ \delta \dfrac{\partial \alpha_0}{\partial \sigma_0}, \delta^3 \dfrac{\partial^3 \alpha_0}{\partial \sigma_0^3}, \cdots \delta^{2n-1} \dfrac{\partial^{2n-1} \alpha_0}{\partial \sigma_0^{2n-1}}, M_1 \right\}. \end{cases}$$

where α_0 and σ_0 are the angle and arc length along the body. These are obtainable by the calculation of $p(\delta), \cdots p^{(n-1)}(\delta), p^{(n)}(\delta)$. The actual determination of δ(or $\delta(\partial\alpha_0/\partial\sigma_0)$) on the right hand side will require another relation.

The calculations by Dugundji include second order terms only. They agree with experimental results, indicating the applicability of the method. Further work seems to be desirable for the calculation of quantities of higher order.

7. Iso-energetic flow in the neighborhood of the nose. In this section, we outline the calculations of the flow conditions in the neighborhood of the nose, which is needed for the discussion of §6.

Although the flow is subsonic, the conditions of flow at the nose can be treated locally, at least to a certain extent. The hydrodynamic quantities, pressure, density, and velocity, must satisfy the equations of motion and the equation of continuity. Furthermore, if the flow is that behind a shock, the Bernoulli function is a constant (iso-energetic), even though the motion is not irrotational. These furnish four relations for the determination of the four hydrodynamic quantities.

If one refers the pressure and density to the stagnation quantities p_0 and ρ_0 and the velocity to $\sqrt{p_0/\rho_0}$, these equations are

$$u \frac{\partial \tau}{\partial x} + v \frac{\partial \tau}{\partial y} = \tau \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \epsilon \frac{v}{y} \right), \tag{7.1}$$

$$u(\partial u/\partial x) + v(\partial u/\partial y) = -\tau(\partial p/\partial x), \tag{7.2}$$

$$u(\partial v/\partial x) + v(\partial v/\partial y) = -\tau(\partial p/\partial y), \tag{7.3}$$

$$p\tau + [(\gamma - 1)/2\gamma](u^2 + v^2) = 1 \tag{7.4}$$

where τ is the specific volume, $\epsilon = 0$ for two dimensional flows, and $\epsilon = 1$ for flows with axial symmetry. In the latter case, y stands for the radial distance from the axis of symmetry. In the two-dimensional case, the flow shall also be assumed to be symmetrical. The shape of the nose may be specified as

$$y^2 = g(x) \tag{7.5}$$

in either case.

We shall take the nose to be the origin of our coordinate system, so that we shall be concerned with small values of x and y. Taking into account the symmetry nature of the flow, we write

$$
(7.6)\qquad \begin{cases} p = 1 + p_1 x + (p_{11}x^2 + p_{22}y^2) + (p_{111}x^3 + p_{122}xy^2) + \cdots, \\ \tau = 1 + \tau_1 x + (\tau_{11}x^2 + \tau_{22}y^2) + (\tau_{111}x^3 + \tau_{122}xy^2) + \cdots, \\ u = u_1 x + (u_{11}x^2 + u_{22}y^2) + \cdots, \\ v = v_1 y + v_{12}xy + \cdots. \end{cases}
$$

By putting these expressions into the equations (7.1) and (7.4), one obtains a series of relations among the coefficients. In particular, one obtains

$$
(7.7)\qquad p_1 = \tau_1 = 0
$$

from (7.2) and (7.4). The other relations, after making use of (7.7), are as follows.

From the equation of continuity (7.1), one obtains

$$
(7.8)\qquad \begin{cases} u_1 + (1+\epsilon)v_2 = 0, \\ 2u_{11} + (1+\epsilon)v_{12} = 0, \\ \cdots \end{cases}
$$

after (7.7) has been used. Equation (7.2) leads to

$$
(7.9)\qquad \begin{cases} -2p_{11} = u_1^2, \\ -p_{111} = u_1 u_{11}, \\ -p_{122} = (u_1 + 2v_2)u_{22}, \\ \cdots. \end{cases}
$$

equations (7.3) leads to

$$
(7.10)\qquad \begin{cases} -2p_{22} = v_2^2 \\ -2p_{122} = (u_1 + 2v_2)v_{12}, \\ \cdots, \end{cases}
$$

while equation (7.4) gives, besides $\tau_1 = 0$,

$$
(7.11)\qquad \begin{cases} p_{11} + \tau_{11} = -\dfrac{\gamma-1}{2\gamma}\, u_1^2, \\ p_{22} + \tau_{22} = -\dfrac{\gamma-1}{2\gamma}\, v_2^2, \\ p_{111} + \tau_{111} = -\dfrac{\gamma-1}{2\gamma}\, u_1 u_{11}, \\ p_{122} + \tau_{122} = -\dfrac{\gamma-1}{\gamma}\, (u_1 u_{22} + v_{12} v_2), \\ \cdots, \end{cases}
$$

In addition, on the boundary (7.5), we must have

$$v/u = dy/dx \tag{7.12}$$

Hence, if one writes

$$y^2 = (2x/\kappa_0) + \cdots \tag{7.13}$$

where κ_0 is the curvature of the nose, there is obtained

$$\begin{cases} (u_1 - 2v_2)k_0/2 + u_{22} = 0, \\ \cdots \end{cases} \tag{7.14}$$

Higher approximations in (7.8)–(7.11) and (7.14) can be easily written down in a similar manner.

The change of entropy. If one now considers the change of entropy, by using the relation

$$p\tau^\gamma = e^{S/c_v} = e^{f(\psi)} \tag{7.15}$$

and expands f as a power series

$$f = s_1 x + (s_{11}x^2 + s_{22}y^2) + (s_{111}x^3 + s_{122}xy^2) + \cdots \tag{7.16}$$

it can be easily shown that,

$$s_1 = s_{11} = s_{22} = s_{111} = 0,$$

after using (7.9)–(7.11). On the other hand, for p_{122}, τ_{122}, and s_{122}, one obtains the relation

$$p_{122} + \gamma\tau_{122} = s_{122} \tag{7.17}$$

As the expressions for p_{122} in (7.9) and (7.10) depend on $u_1 + 2v_2$, which vanishes in the axially symmetrical case but not in the two-dimensional case (cf. (7.8)), the exact relations depend on the particular case we are concerned with.

Case 1. Two dimensional case. Here

$$u_1 + 2v_2 \neq 0,$$

and hence (7.9) and (7.10) give

$$v_{12} = 2u_{22} \tag{7.18}$$

Putting this expression in (7.11), one obtains

$$p_{122} + \tau_{122} = -[(\gamma - 1)/\gamma](u_1 + 2v_2)u_{22}.$$

Making use of (7.9) again, this gives

$$p_{122} = -\gamma\tau_{122},$$

and hence (7.17) gives

$$s_{122} = 0.$$

Thus, in this case

$$S/c_v = f = s_{1111}x^4 + s_{1122}x^2y^2 + s_{2222}y^4 + \cdots. \tag{7.19}$$

Case 2. Axially symmetrical case. Here

$$u_1 + 2v_2 = 0,$$

and hence (7.9) and (7.10) lead to the identical result

$$p_{122} = 0. \tag{7.20}$$

Eqs. (7.11) and (7.17) then lead to

$$\tau_{122} = -[(\gamma - 1)/\gamma](u_1u_{22} + v_{12}v_2) \tag{7.21}$$

and

$$s_{122} = -(\gamma - 1)(u_1u_{22} + v_{12}v_2), \tag{7.22}$$

so that

$$S/c_v = s_{122}xy^2 + \cdots. \tag{7.23}$$

The difference between these two cases can be understood more clearly by the following consideration. In either case,

$$S/c_v = y^2\varphi(x,y)$$

by symmetry. Also,

$$S/c_v = f(\psi)$$

where ψ is the stream function. Now, in the two-dimensional case,

$$\psi = u_1xy + \cdots.$$

Hence, in order that $S \sim y^2$ one must have

$$S \sim \psi^2 \sim x^2y^2 \tag{7.24}$$

In the axially symmetrical case, we have the Stokes stream function

$$\psi = \tfrac{1}{2}u_1xy^2 + \cdots.$$

Hence,

$$S \sim \psi \sim xy^2. \tag{7.25}$$

It is now also clear that

$$s_{122} = (df/d\psi)_0u_1/2. \tag{7.26}$$

The quantity $(df/d\psi)_0$ can be calculated at any point in the field of flow. In particular, it can be most conveniently calculated at the shock, where the change of entropy is produced.

Summarizing the above results, we find that, in the two-dimensional case,

$$
(7.27)\qquad \begin{cases} p = 1 - \tfrac{1}{2}a^2(x^2 + y^2) - \dfrac{3a^2}{2}\kappa_0(x^3 + xy^2) + \cdots, \\ \tau = 1 + \dfrac{1}{2\gamma}a^2(x^2 + y^2) + \dfrac{3a^2}{2\gamma}\kappa_0(x^3 + \tfrac{1}{3}xy^2) + \cdots, \\ u = a\left\{x + \dfrac{3\kappa_0}{2}(x^2 - y^2) + \cdots\right\}, \\ v = -a\{y + 3\kappa_0 xy + \cdots\}, \end{cases}
$$

which depend on a single parameter a. In the axially symmetrical case,

$$
(7.28)\qquad \begin{cases} p = 1 - \tfrac{1}{2}a^2(x^2 + \tfrac{1}{4}y^2) - abx^3 + \cdots \\ \tau = 1 + \dfrac{1}{2\gamma}a^2(x^2 + \tfrac{1}{4}y^2) + \dfrac{1}{\gamma}abx^3 + \dfrac{\gamma - 1}{\gamma}\left(a^2\kappa_0 - \dfrac{ab}{2}\right)xy^2 + \cdots \\ S/c_v = f = (\gamma - 1)(a_2\kappa_0 - \tfrac{1}{2}ab)xy^2 + \cdots \\ u = ax + bx^2 - a\kappa_0 y^2 + \cdots \\ v = -\tfrac{1}{2}ay - bxy + \cdots \end{cases}
$$

which depend on two parameters a and b. In addition, the condition

$$
(7.29)\qquad (df/d\psi)_0 = 2s_{122}/u_1 = 2(\gamma - 1)(a\kappa_0 - \tfrac{1}{2}b)
$$

must be satisfied. Hence, if $(df/d\psi)_0$ is calculated, there is again only a single parameter, since

$$
(7.30)\qquad b = 2a\kappa_0 - [(df/d\psi)_0/(\gamma - 1)]
$$

It is interesting to note the fact that $(df/d\psi)_0$ vanishes in the two-dimensional case, but does not vanish in the axially symmetrical case. This causes a difference between the two cases in the relation of the sonic line with respect to a straight solid boundary. Over a straight boundary, $\partial\theta/\partial s = 0$ and hence (2.15) and (2.17) give

$$
(7.31)\qquad \frac{\partial p}{\partial n} = 0, \qquad \frac{\partial q}{\partial n} = \omega = \frac{p}{R}\frac{dS}{d\psi}y'.
$$

In the two-dimensional case, $(\partial S/\partial\psi)_0 = 0$, and we have

$$
(7.32)\qquad \partial q/\partial n = 0,
$$

and hence (2.9) gives

$$
(7.33)\qquad \partial\rho/\partial n = 0.
$$

Thus, the normal to the boundary is locally a line of constant p, q, ρ, and in particular the sonic line in perpendicular to the straight boundary in the two-dimensional case. In the axially symmetrical case, the perpendicular line is not a line of constant velocity.

REFERENCES

1. Ladenburg, R., Van Voorhis, C. C., and Winkler, J., Interferometric Study of supersonic phenomena NAVORD Reports 69-46 (Part I) 93-46 (Part II), 7-47 (Part III).
2. Crocco, L., L'Aerotechnia **17**, 519 (1937).
3. Hentzsche, Lilienthal Gesellschaft Bericht **156**, 174 (1942).
4. Thomas, T. Y., J. Math. and Phys. **26**, 62-68 (1947).
5. Maccoll, J. W., and Codd, J., Theoretical investigation of the flow around various bodies in the sonic region of velocities, Theoretical Research Reports No. 17/45 (Sept. 1945) and No. 7/46 (Sept., 1946).
6. Laitone, E. V., and Pardee, O. O'M., Location of detached shock wave in front of a body moving at supersonic speeds. NACA (~~Restricted~~ Reclassified) RM No. A7B10 (May 6, 1947).
7. Emmons, H. W., The theoretical flow of a frictionless adiabatic perfect gas inside of a two-dimensional hypersonic nozzle. Appendix II, NACA Technical Note No. 1003 (1946).
8. Tsien, H. S., J. Math. and Phys. **26**, 69-75 (1947).
9. Vazsonyi, A., Quarterly of Appl. Math. **3**, 29-37 (1945).
10. Bateman, H., Proc. Nat. Acad. Sci. **16**, 816-825 (1930).

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SOME GEOMETRIC PROPERTIES OF PLANE GAS FLOW

By P. Neményi and R. Prim

This paper deals with steady flow of a perfect gas (i.e., an inviscid, thermally non-conducting gas having constant specific heats) in the absence of extraneous forces. The fundamental equations governing such flows can be most conveniently formulated in terms of the pressure p and the reduced velocity vector $\boldsymbol{w}$ defined in terms of the actual velocity vector $\boldsymbol{v}$ and the ultimate velocity magnitude a by

$$\boldsymbol{w} = \boldsymbol{v}/a.$$

In this formulation the continuity equation takes on the form

$$\text{(C)} \qquad \operatorname{div}[(1 - w^2)^{1/(\gamma-1)}\boldsymbol{w}] = 0$$

where γ denotes the adiabatic exponent, while the dynamic equation can be written

$$\text{(D)} \qquad \boldsymbol{w} \cdot \operatorname{grad} \boldsymbol{w} + (1 - w^2) \operatorname{grad} \ln p^{(\gamma-1)/2\gamma} = 0.$$

Equation (D) implies the following condition on the vector field $\boldsymbol{w}$:

$$\text{(W)} \qquad \operatorname{curl}\left[\frac{\boldsymbol{w} \times \operatorname{curl} \boldsymbol{w}}{1 - w^2}\right] = 0.$$

These differential equations were essentially contained in the work of Crocco[1] but their validity for the general case of non-uniform ultimate velocity was established later by the independent investigations of Munk and Prim[2] and of Hicks, Guenther, and Wasserman.[3]

A natural coordinate system for the analysis of problems of plane fluid motions employs as coordinate curves the streamlines (ξ = const.) and their orthogonal trajectories (η = const.). Denoting by $g_1(\xi, \eta)d\xi$ and $g_2(\xi, \eta)d\eta$ the components of the vector element of arc length in this coordinate system we have the following relations

$$\text{(1)} \qquad \begin{cases} \operatorname{curl} \boldsymbol{w} = -\dfrac{1}{g_1 g_2}\dfrac{\partial}{\partial \xi}(g_2 w) = -\dfrac{w}{g_1}\dfrac{\partial \ln g_2}{\partial \xi} - \dfrac{1}{g_1}\dfrac{\partial w}{\partial \xi} \\ \left[\dfrac{\boldsymbol{w} \times \operatorname{curl} \boldsymbol{w}}{1 - w^2}\right]_\eta = 0 \\ \left[\dfrac{\boldsymbol{w} \times \operatorname{curl} \boldsymbol{w}}{1 - w^2}\right]_\xi = \dfrac{1}{(1 - w^2)g_1}\left[w^2 \dfrac{\partial \ln g_2}{\partial \xi} + \tfrac{1}{2}\dfrac{\partial (w^2)}{\partial \xi}\right]. \end{cases}$$

By equation (W) the vector in the brackets is irrotational. Hence we have the vorticity equation

$$\text{(2)} \qquad \frac{\partial}{\partial \eta}\left\{g_1\left[\frac{\boldsymbol{w} \times \operatorname{curl} \boldsymbol{w}}{1 - w^2}\right]_\eta\right\} = -\frac{\partial}{\partial \eta}\left[\frac{w^2}{1 - w^2}\frac{\partial \ln(g_2 w)}{\partial \xi}\right] = 0$$

or

$$\frac{\partial \ln g_2}{\partial \xi}\frac{\partial}{\partial \eta}\left(\frac{w^2}{w^2-1}\right)+\frac{w^2}{w^2-1}\frac{\partial^2 \ln g_2}{\partial \xi \partial \eta}+\tfrac{1}{2}\frac{\partial \ln(1-w^2)}{\partial \xi \partial \eta}=0$$

As for the continuity equation, (C), it can immediately be written in terms of our natural coordinates as follows:

$$(\partial/\partial\eta)[g_1(1-w^2)^{\beta}w]=0. \tag{3}$$

These two equations represent the only conditions which need be satisfied by a vector $\boldsymbol{w}$ in order for it to be the reduced velocity vector associated with a family of steady plane flows of a perfect gas.

Assume now that w is constant along each streamline. This means that everywhere $\partial w/\partial \eta = 0$. The vorticity equation then becomes a condition for g_2 and the continuity equation a condition for g_1, while both equations leave $w(\xi)$ arbitrary. The conditions for the g'-s are:

$$\partial^2 \ln g_2/\partial\xi\partial\eta = 0 \quad \text{or} \quad g_2 = d(\xi)\cdot b(\eta)$$

and

$$\partial \ln g_1/\partial\eta = 0 \quad \text{or} \quad g_1 = c(\xi) \tag{4}$$

d, b, c denoting arbitrary functions.

There exist various methods for obtaining the equations of the curve system from the functions g_1 and g_2. We will follow the method which was used—in an entirely different connection—by W. Tollmien[4].

First of all, the requirement that the $\xi-\eta$ net lie in a plane is expressed by the Gaussian theorem.

$$\frac{\partial}{\partial\xi}\left[\frac{1}{g_1}\frac{\partial g_2}{\partial\xi}\right]+\frac{\partial}{\partial\eta}\left[\frac{1}{g_2}\frac{\partial g_1}{\partial\eta}\right]=0.$$

This yields a connection between the two arbitrary functions $d(\xi)$ and $c(\xi)$:

$$d'(\xi) = Cc(\xi)$$

where C is an arbitrary constant. We represent g_1 and g_2 in terms of the complex variable $z = x + iy$

$$g_1 = |\,\partial z/\partial\xi\,| \qquad g_2 = |\,\partial z/\partial\eta\,|$$

Hence

$$\partial z/\partial\xi = g_1 e^{iK_1} \qquad \partial z/\partial\eta = g_2 e^{iK_2} \tag{5}$$

K_1, K_2 being real functions of ξ, η, the geometric meaning of which is the angle between the tangent to the coordinate curves and the positive direction of the x-axis. The orthogonality of our curvilinear net implies that $\sin(K_2 - K_1) = \pm 1$ or

$$e^{iK_2} = \pm ie^{iK_1}. \tag{6}$$

From (5) follows by cross differentiation

$$e^{iK_1}\left[\frac{\partial g_1}{\partial \eta} + ig_1\frac{\partial K_1}{\partial \eta}\right] = e^{iK_2}\left[\frac{\partial g_2}{\partial \xi} + ig_2\frac{\partial K_2}{\partial \xi}\right].$$

Hence taking into account relation (6) we obtain

$$\partial g_1/\partial \eta = \mp g_2\,\partial K_2/\partial \xi$$

and

$$\partial g_2/\partial \xi = \pm g_1\,\partial K_1/\partial \eta.$$

Upon substitution of the expressions (4) for g_1 and g_2 we obtain

$$\frac{\partial K_2}{\partial \xi} = \mp\frac{1}{g_2}\frac{\partial g_1}{\partial \eta} = 0 \quad \text{or} \quad K_2 = e(\eta)$$

and

$$\frac{\partial K_1}{\partial \eta} = \pm\frac{1}{g_1}\frac{\partial g_2}{\partial \xi} = \pm\frac{C}{c'(\xi)}\,b(\eta)\,d'(\xi) = \pm Cb(\eta).$$

However, from (6)

$$K_1 = K_2 \mp \pi/2 = e(\eta) \mp \pi/2$$

and therefore

$$\partial K_1/\partial \eta = e'(\eta) = \pm\, Cb(\eta).$$

We distinguish two cases.

Case I. $C \neq 0$

$$\partial z/\partial \xi = (d'(\xi)/C)e^{i[e(\eta)\mp\pi/2]} = \mp\, i(d'(\xi)/C)e^{ie(\eta)}$$

$$\partial z/\partial \eta = \pm\,(d(\xi)/C)e'(\eta)e^{ie(\eta)}.$$

The solution of this system of simultaneous equations is

$$z = \mp\,(d(\xi)/C)e^{ie(\eta)} + F + iG$$

where F and G are real constants.
Hence

$$x = +\,d(\xi)\sin e(\eta) + F$$

$$y = -\,d(\xi)\cos e(\eta) + G$$

where $\pm C$ has been included in the arbitrary functions $d(\xi)$. This represents an arbitrary family of concentric circles.

Case II. $C = 0$. In this case

$$e'(\eta) = 0 \text{ and } e(\eta) = E = \text{constant}$$

$$d'(\xi) = 0 \text{ and } d(\xi) = A = \text{constant}.$$

Hence we have

$$\partial z/\partial \xi = \mp\, ic(\xi)e^{iE}$$

$$\partial z/\partial \eta = Ab(\eta)e^{iE}$$

for which the solution is

$$z = \mp ie^{iE} \int c(\xi)\, d\xi + Ae^{iE} \int b(\eta)\, d\eta.$$

Therefore

$$x = \pm \sin E \int c(\xi)\, d\xi + A \cos E \int b(\eta)\, d\eta$$

$$y = \mp \cos E \int c(\xi)\, d\xi + A \sin E \int b(\eta)\, d\eta.$$

In this case the streamlines are parallel straight lines. (The arbitrary functions indicate merely arbitrariness of parametrization).

Since these two cases exhaust all possibilities we have obtained

THEOREM 1. *The reduced velocity magnitude, and hence also the Mach number, can have constant values along each streamline only if all streamlines are concentric circles or parallel straight lines.* The same relation is valid for the actual velocity field because the ultimate velocity magnitude has a constant value along each streamline.

In this respect rotational compressible flow is subject to the same type of geometrical restriction which is well known for irrotational incompressible flow. However, while in the latter case the function $w(\xi)$ must be proportional to the reciprocal of the radius or, in the case of straight streamlines, a constant, in the present case $w(\xi)$ can be any function.

It is obvious that in all these cases in which w = const. and hence also v = const. along each streamline, curl $\boldsymbol{w}$ and curl $\boldsymbol{v}$ also have a constant magnitude along the same lines. The question arises, however, whether the inverse theorem is also true, that is, whether the only plane flow patterns possible in a perfect gas such that the streamlines are lines of equal vorticity, are those in which they are also lines of equal velocity magnitude. This question is interesting because it is well known* that in incompressible flow (viscous or inviscid) there exist, outside the cases of flows with concentric circles and with parallel straight line as streamlines, large families of steady flow for which the vorticity is constant along each streamline. To settle the corresponding question for the flow of perfect gases, let us express the curl of the actual velocity field $\boldsymbol{v}$ in terms of the reduced velocity field $\boldsymbol{w}$. From curl $\boldsymbol{w}$ = curl $(\boldsymbol{v}/a)$ (where a is the "ultimate velocity" magnitude) we obtain for plane flow fields by a simple vector transformation:

$$|\operatorname{curl} \boldsymbol{w}| = (1/a)[|\operatorname{curl} \boldsymbol{v}| + v\,|\operatorname{grad} \ln a|]$$

and hence

$$|\operatorname{curl} \boldsymbol{v}| = a[|\operatorname{curl} \boldsymbol{w}| - (w/g_1)d(\ln a)/d\xi]. \tag{7}$$

* See especially the investigations of J. Kampé de Feriet quoted and discussed by Ratib Berker: "Sur quelques cas d'integration des équations du mouvement d'un fluid visqueux incompressible." Paris-Lille, 1936.

Now $d(\ln a)/d\xi$ depends only on ξ, a being constant along each streamline. The condition of constant actual vorticity along a streamline is expressed therefore by

$$(\partial/\partial\eta)|\operatorname{curl} \boldsymbol{v}| = a(\partial/\partial\eta)|\operatorname{curl} \boldsymbol{w}| + af(\xi)(\partial/\partial\eta)(w/g_1) = 0$$

or

$$|\operatorname{curl} \boldsymbol{w}| + f(\xi)(w/g_1) = h(\xi) = \text{an arbitrary function of } \xi.$$

Hence

$$(1/g_1g_2)\partial(g_2w)/\partial\xi = h(\xi) - f(\xi)w/g_1 .$$

Substituting this in our vorticity equation we obtain

$$\frac{\partial}{\partial\eta}\left[\frac{wg_1}{1-w^2}\,h(\xi) - f(\xi)\,\frac{w^2}{1-w^2}\right] = 0.$$

However, from the continuity equation it follows that

$$g_1w = K(\xi)(1-w^2)^{-\beta}.$$

Substituting this into the first term within the brackets of (4) we obtain

$$h(\xi)K(\xi)(\partial/\partial\eta)[(1-w^2)^{-\beta-1}] = f(\xi)(\partial/\partial\eta)[w^2(1-w^2)^{-1}].$$

As in a rotational flow it is impossible to have both $f(\xi)$ and one of the functions $h(\xi)$ or $K(\xi)$ identically zero, it follows that $\partial w/\partial\eta = 0$ and we have proved:

THEOREM 2. *In plane steady flow of a perfect gas constant vorticity along each streamline implies constant velocity along the same lines.*

As a corollary, we see that the only plane steady flows of a perfect gas for which the vorticity is constant along each streamline are those whose streamlines are concentric circles or parallel straight lines.

By similar, but slightly simpler, argument the following related theorem can be proved:

THEOREM 2A. *In plane steady flow of a perfect gas constant vorticity of the reduced velocity field along each streamline implies constant reduced velocity along the same lines.*

A further, closely related question is the following: Under what conditions in plane flow of a perfect gas are the lines of equal vorticity ("isocurls") identical with the lines of equal velocity magnitude ("isovels")? This relation can be expressed by the equation

$$|\operatorname{curl} \boldsymbol{v}| = j(v).$$

According to equation (7) this can be written in terms of the reduced velocity $\boldsymbol{w}$ and of the ultimate velocity magnitude a (which is always a function of ξ alone) as follows:

$$a(\xi)\,|\operatorname{curl} \boldsymbol{w}| + (w/g_1)\,d[a(\xi)]/d\xi = j[a(\xi)w]$$

or, by virtue of equation (1)

$$\frac{a(\xi)}{g_1g_2}\frac{\partial}{\partial\xi}(g_2w) + \frac{w}{g_1}\frac{da(\xi)}{d\xi} = j[a(\xi)w]$$

or

$$\frac{\partial}{\partial \xi} \ln(g_2 w) + \frac{d \ln a(\xi)}{d\xi} = \frac{j[a(\xi)w]g_1}{a(\xi)w}. \tag{8}$$

However, from (3) we have

$$g_1 = m(\xi)/w(1 - w^2)^\beta$$

and from (2)

$$\frac{\partial}{\partial \xi} \ln(g_2 w) = [(1 - w^2)/w^2]n(\xi),$$

where m and n are arbitrary functions. Substituting these expressions into equation (8) we obtain

$$\frac{1 - w^2}{w^2} n(\xi) + \frac{d \ln a(\xi)}{d\xi} = \frac{j[a(\xi)w]}{a(\xi)\, w} \cdot \frac{m(\xi)}{w(1 - w^2)^\beta}.$$

Since this equation involves only w and functions of ξ it follows that w is independent of η and we have

THEOREM 3. *In plane rotational flow of a perfect gas the isovels and isocurls of the velocity field can coincide only if they are streamlines.*

From this follows easily:

THEOREM 3A. *In plane rotational flow of a perfect gas the isovels and isocurls of the reduced velocity field can coincide only if they are streamlines.*

It will be noted that in all problems discussed here the geometric limitations of plane rotational gas flow are just as restrictive as those of plane irrotational gas flow.

REFERENCES

1) L. Crocco, *Eine neue Stromfunktion für die Erforschung der Bewegung der Gase mit Rotation*, Z. f. Angewandte Math. und Mech. **17**, 1–7 (1937).

2) M. M. Munk and R. C Prim, *On the Canonical Form of the Equations of Steady Motion of a Perfect Gas*, Naval Ordnance Laboratory Memorandum 9219, June 1947.

3) B. Hicks, P. Guenther, and P. Wasserman, *New Formulations of the Equations for Compressible Flow*, Quarterly of Applied Math. **5**, 357–361 (1947).

4) W. Tollmien, *Zum Übergang von Unterschall—in Überschall-Strömungen*, Z. f. Angewandte Mathematik und Mechanik **17**, 117–136 (1937).

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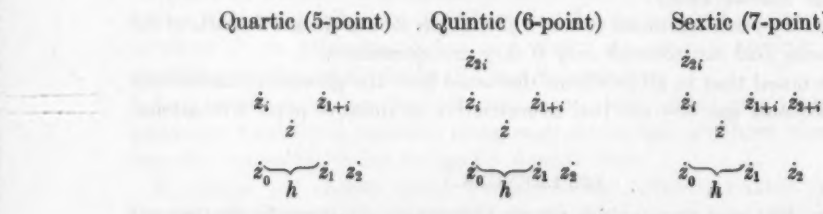
COEFFICIENTS FOR COMPLEX QUARTIC, QUINTIC, AND SEXTIC INTERPOLATION WITHIN A SQUARE GRID

By Herbert E. Salzer

The present table of Lagrangian interpolation coefficients for an analytic function tabulated over a square grid in the complex plane, is an extension of a previous table that gave 3-point and 4-point coefficients.[1] Here coefficients are given for the 5-, 6-, and 7-point cases (i.e. when the function $f(z)$ can be approximated by a quartic, quintic, or sextic in z). There was considerable latitude in the choice of the fixed points z_ν of the square grid, upon which to base the approximating polynomials. The deciding factor was the location of the points z_ν to be a compromise between being as closely together as possible, and at the same time allowing the point z to be as "central as possible" with respect to all the points z_ν. Both these factors were limited by the choice of all points z_ν to lie within a region with an L-shaped boundary. This last consideration enables these coefficients to be used anywhere in the plane, including the beginning, end, or any combination of beginning or end, with respect to either x or y, of a table in $z = x + iy$. The points z_ν were chosen according to the scheme below:[2]



The n-point interpolation formulas for $f(z) \equiv f(z_0 + Ph)$, where h denotes the length of a side of the square comprising the grid, and $P = p + iq$ denotes the complex tabular fraction, are of the form $\sum L_\nu^{(n-1)}(P)f(z_\nu)$, where the summation is over the points indicated in the above diagrams for $n = 5$, 6, and 7. The explicit expressions for $L_\nu^{(n-1)}(P)$ are as follows:

C. Quartic (5-point Interpolation)

$$f(z_0 + Ph) = L_0^{(4)}(P)f(z_0) + L_1^{(4)}(P)f(z_1) + L_2^{(4)}(P)f(z_2) + L_i^{(4)}(P)f(z_i) + L_{1+i}^{(4)}(P)f(z_{1+i}), \text{ where}$$

$$L_0^{(4)}(P) = \tfrac{1}{6}[(-1-i)P^4 + (2+6i)P^3 + (3-11i)P^2 + (-8+6i)P + 4]$$

[1] A. N. Lowan, H. E. Salzer, "Coefficients for Interpolation within a Square Grid in the Complex Plane", Jour. of Math. and Physics, vol. 23, 1944, pp. 156–166.

[2] Note the change in notation, so that z_{-1} and z_2 of the previous paper correspond to z_i and z_{1+i} respectively; likewise the previous $L_{-1}^{(3)}(P)$ would read $L_i^{(3)}(P)$, etc. In consistency with the notation of the previous paper, the superscript m in $L_\nu^{(m)}(P)$ denotes its degree in P, rather than the number of points, which is $m + 1$.

$$L_1^{(4)}(P) = \tfrac{1}{2}[(1 - i)P^4 + (-5 + i)P^3 + (6 + 4i)P^2 + (-4i)P]$$

$$L_2^{(4)}(P) = \tfrac{1}{30}[(1 + 3i)P^4 + (4 - 8i)P^3 + (-9 + 3i)P^2 + (4 + 2i)P]$$

$$L_i^{(4)}(P) = \tfrac{1}{10}[(-3 + i)P^4 + (13 - i)P^3 + (-18 - 4i)P^2 + (8 + 4i)P]$$

$$L_{1+i}^{(4)}(P) = \tfrac{1}{2}[iP^4 + (1 - 3i)P^3 + (-3 + 2i)P^2 + 2P]$$

D. Quintic (6-point Interpolation)

$$f(z_0 + Ph) = L_0^{(5)}(P)f(z_0) + L_1^{(5)}(P)f(z_1) + L_2^{(5)}(P)f(z_2) + L_i^{(5)}(P)f(z_i) + L_{1+i}^{(5)}(P)f(z_{1+i}) + L_{2i}^{(5)}(P)f(z_{2i}), \text{ where}$$

$$L_0^{(5)}(P) = \tfrac{1}{8}[(1 - i)P^5 - 8P^4 + (15 + 15i)P^3 + (-30i)P^2 + (-16 + 16i)P + 8]$$

$$L_1^{(5)}(P) = \tfrac{1}{10}[(3 + i)P^5 + (-5 - 15i)P^4 + (-20 + 30i)P^3 + 40P^2 + (-8 - 16i)P]$$

$$L_2^{(5)}(P) = \tfrac{1}{40}[(-1 + 2i)P^5 + 10P^4 + (-10 - 15i)P^3 + (-5 + 15i)P^2 + (6 - 2i)P]$$

$$L_i^{(5)}(P) = \tfrac{1}{10}[(-1 - 3i)P^5 + (-5 + 15i)P^4 + (30 - 20i)P^3 - 40P^2 + (16 + 8i)P]$$

$$L_{1+i}^{(5)}(P) = \tfrac{1}{4}[(-1 + i)P^5 + 6P^4 + (-9 - 9i)P^3 + (12i)P^2 + (4 - 4i)P]$$

$$L_{2i}^{(5)}(P) = \tfrac{1}{40}[(-2 + i)P^5 + 10P^4 + (-15 - 10i)P^3 + (5 + 15i)P^2 + (2 - 6i)P]$$

E. Sextic (7-point Interpolation)

$$f(z_0 + Ph) = L_0^{(6)}(P)f(z_0) + L_1^{(6)}(P)f(z_1) + L_2^{(6)}(P)f(z_2) + L_i^{(6)}(P)f(z_i) + L_{1+i}^{(6)}(P)f(z_{1+i}) + L_{2+i}^{(6)}(P)f(z_{2+i}) + L_{2i}^{(6)}(P)f(z_{2i}), \text{ where}$$

$$L_0^{(6)}(P) = \tfrac{1}{40}[(-1 + 3i)P^6 + (21 - 13i)P^5 + (-85 - 15i)P^4 + (105 + 135i)P^3 + (16 - 198i)P^2 + (-96 + 88i)P + 40]$$

$$L_1^{(6)}(P) = \tfrac{1}{10}[(-2 + i)P^6 + (15 + 5i)P^5 + (-20 - 45i)P^4 + (-35 + 75i)P^3 + (72 - 16i)P^2 + (-20 - 20i)P]$$

$$L_2^{(6)}(P) = \tfrac{1}{40}[(-2 - i)P^6 + (3 + 14i)P^5 + (25 - 30i)P^4 - 55P^3 + (27 + 31i)P^2 + (2 - 14i)P]$$

$$L_i^{(6)}(P) = \tfrac{1}{30}[(1 + 3i)P^6 + (6 - 22i)P^5 + (-55 + 45i)P^4 + (120 - 10i)P^3 + (-96 - 48i)P^2 + (24 + 32i)P]$$

$$L_{1+i}^{(6)}(P) = \tfrac{1}{4}[(1-i)P^6 + (-9+i)P^5 + (21+15i)P^4 + (-9-39i)P^3 + (-16+28i)P^2 + (12-4i)P]$$

$$L_{2+i}^{(6)}(P) = \tfrac{1}{720}[(-1-i)P^6 + (8i)P^5 + (15-15i)P^4 - 30P^3 + (16+16i)P^2 + (-8i)P]$$

$$L_{2i}^{(6)}(P) = \tfrac{1}{240}[P^6 + (-6-3i)P^5 + (10+15i)P^4 + (-25i)P^3 + (-11+15i)P^2 + (6-2i)P]$$

All of the above functions $L_\nu^{(n-1)}(P)$ are here tabulated exactly for p and q both ranging independently from 0.0 to 1.0 at intervals of 0.1.

Inverse interpolation can be performed according to a suitable modification of a formula due to the author.[3] The problem is to find P when $f(z) \equiv f(z_0 + Ph)$ is given. When n-point direct interpolation suffices, consider the polynomials $L_\nu^{(n-1)}(P)$ which are given above. Let the complex coefficients of P, P^2, $\cdots$, P^6 be denoted by $A_\nu^{(n-1)}$ $B_\nu^{(n-1)}$, $\cdots$, $F_\nu^{(n-1)}$ respectively. Then if $D = \sum A_\nu^{(n-1)} f(z_\nu), r = [f(z) - f(z_0)]/D, s = [\sum B_\nu^{(n-1)} f(z_\nu)]/D, t = [\sum C_\nu^{(n-1)} f(z_\nu)]/D$ $\cdots$, $w = [\sum F_\nu^{(n-1)} f(z_\nu)]/D$, where all summations in ν are over the points in the above diagrams, the quantity P is given by the following formula:

$$P = r - r^2 s + r^3(2s^2 - t) + r^4(-5s^3 + 5st - u) + r^5(14s^4 - 21s^2 t + 3t^2 + 6su - v) + r^6(-42s^5 + 84s^3 t - 28st^2 - 28s^2 u + 7tu + 7sv - w) + \cdots$$

In actual practise, a large number of terms in this expression for P will not be needed, due either to their absence when $n < 7$, or to their smallness when they do occur.

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[3] "A New Formula for Inverse Interpolation", Bull. Amer. Math. Soc., **50**, 1944, pp 513–516.

TABLE OF COEFFICIENTS: 5-POINT
$L_0^{(4)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	1.000000 +.000000i	.807975 +.123975i	.633600 +.201600i	.478975 +.240975i	.345600 +.249600i	.234375 +.234375i	.145600 +.201600i	.078975 +.156975i	.033600 +.105600i	.007975 +.051975i	.000000 +.000000i
0.1	.843975 −.173025i	.701100 −.036900i	.567775 +.055575i	.446700 +.111900i	.339975 +.138975i	.249100 +.143100i	.174975 +.129975i	.117900 +.104700i	.077575 +.071775i	.053100 +.035100i	.042975 −.002025i
0.2	.681600 −.294400i	.584575 −.154225i	.489600 −.054400i	.399975 +.011975i	.318400 +.051200i	.246975 +.068975i	.187200 +.070400i	.139975 +.059975i	.105600 +.041600i	.083775 +.018575i	.073600 −.006400i
0.3	.520975 −.368025i	.465900 −.231300i	.405975 −.131025i	.345100 −.060900i	.286575 −.015225i	.233100 +.011100i	.186775 +.022575i	.149100 +.023100i	.120975 +.015975i	.102700 +.003900i	.093975 −.011025i
0.4	.369600 −.398400i	.351975 −.272025i	.323200 −.177600i	.287775 −.109425i	.249600 −.062400i	.211975 −.032025i	.177600 −.014400i	.148575 −.006225i	.126400 −.004800i	.111975 −.008025i	.105600 −.014400i
0.5	.234375 −.390625i	.249100 −.280900i	.246975 −.198025i	.233100 −.136900i	.211975 −.093025i	.187500 −.062500i	.162975 −.042025i	.141100 −.028900i	.123975 −.021025i	.113100 −.016900i	.109375 −.015625i
0.6	.121600 −.350400i	.162975 −.263025i	.182400 −.196800i	.185575 −.147225i	.177600 −.110400i	.162975 −.083025i	.145600 −.062400i	.128775 −.046425i	.115200 −.033600i	.106975 −.023025i	.105600 −.014400i
0.7	.036975 −.284025i	.098700 −.224100i	.133975 −.179025i	.149100 −.144900i	.149775 −.118425i	.141100 −.096900i	.127575 −.078225i	.113100 −.060900i	.100975 −.044025i	.093900 −.027300i	.093975 −.011025i
0.8	−.014400 −.198400i	.060775 −.170425i	.105600 −.150400i	.126975 −.135025i	.131200 −.121600i	.123975 −.108025i	.110400 −.092800i	.094975 −.075025i	.081600 −.054400i	.073575 −.031225i	.073600 −.006400i
0.9	−.028025 −.101025i	.053100 −.108900i	.100575 −.117225i	.121900 −.123300i	.123975 −.125025i	.113100 −.120900i	.094975 −.110025i	.074700 −.092100i	.056775 −.067425i	.045100 −.036900i	.042975 −.002025i
1.0	.000000 +.000000i	.078975 −.047025i	.121600 −.086400i	.135975 −.116025i	.129600 −.134400i	.109375 −.140625i	.081600 −.134400i	.051975 −.116025i	.025600 −.086400i	.006975 −.047025i	.000000 +.000000i

TABLE OF COEFFICIENTS: 5-POINT

$L_4^{(0)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	.00000 +.00000i	.02755 −.17955i	.10080 −.31680i	.20655 −.41055i	.33280 −.46080i	.46875 −.46875i	.60480 −.43680i	.73255 −.36855i	.84480 −.26880i	.93555 −.14355i	1.00000 +.00000i
0.1	.17055 −.01755i	.16380 −.14580i	.20035 −.24495i	.26820 −.31260i	.35655 −.34755i	.45580 −.34980i	.55755 −.32055i	.65460 −.26220i	.74095 −.17835i	.81180 −.07380i	.86355 +.04545i
0.2	.28480 −.06080i	.25695 −.14135i	.26880 −.20480i	.30955 −.24755i	.36960 −.26720i	.44055 −.26255i	.51520 −.23360i	.58755 −.18155i	.65280 −.10880i	.70735 −.01895i	.74880 +.08320i
0.3	.34755 −.11655i	.31060 −.15420i	.30855 −.18555i	.33180 −.20580i	.37195 −.21135i	.42180 −.19980i	.47535 −.16995i	.52780 −.12180i	.57555 −.05655i	.61620 +.02340i	.64855 +.11445i
0.4	.36480 −.17280i	.32955 −.17355i	.32320 −.17760i	.33735 −.17895i	.36480 −.17280i	.39955 −.15555i	.43680 −.12480i	.47295 −.07935i	.50560 −.01920i	.53355 +.05445i	.55680 +.13920i
0.5	.34375 −.21875i	.31980 −.18980i	.31755 −.17255i	.32980 −.15980i	.35055 −.14555i	.37500 −.12500i	.39955 −.09455i	.42180 −.05180i	.44055 +.00445i	.45580 +.07420i	.46875 +.15625i
0.6	.29280 −.24480i	.28855 −.19455i	.29760 −.16320i	.31395 −.14235i	.33280 −.12480i	.35055 −.10455i	.36480 −.07680i	.37435 −.03795i	.37920 +.01440i	.38055 +.08145i	.38080 +.16320i
0.7	.22155 −.24255i	.24420 −.18060i	.27055 −.14355i	.29580 −.12180i	.31635 −.10695i	.32980 −.09180i	.33495 −.07035i	.33180 −.03780i	.32155 +.00945i	.30660 +.07380i	.29055 +.15645i
0.8	.14080 −.20480i	.19635 −.14195i	.24480 −.10880i	.28255 −.09455i	.30720 −.08960i	.31755 −.08555i	.31360 −.07520i	.29655 −.05255i	.26880 −.01280i	.23395 +.04765i	.19680 +.13120i
0.9	.06255 −.12555i	.15580 −.07380i	.22995 −.05535i	.28260 −.05820i	.31255 −.07155i	.31980 −.08580i	.30555 −.09255i	.27220 −.08460i	.22335 −.05595i	.16380 −.00180i	.09955 +.08145i
1.0	.00000 +.00000i	.13455 +.02745i	.23680 +.01920i	.30555 −.01155i	.34080 −.05280i	.34375 −.09375i	.31680 −.12480i	.26355 −.13755i	.18880 −.12480i	.09855 −.08055i	.00000 +.00000i

TABLE OF COEFFICIENTS: 5-POINT

$L_2^{(0)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	.000000 +.000000i	.015705 +.011115i	.023680 +.023040i	.025305 +.033915i	.022080 +.042240i	.015625 +.046875i	.007680 +.047040i	.000105 +.042315i	−.005120 +.032640i	−.005895 +.018315i	.000000 +.000000i
0.1	−.005895 +.018315i	.007380 +.022140i	.014905 +.027915i	.017700 +.033900i	.016905 +.038715i	.013780 +.041340i	.009705 +.041115i	.006180 +.037740i	.004825 +.031275i	.007380 +.022140i	.015705 +.011115i
0.2	−.005120 +.032640i	.004825 +.031275i	.010880 +.032640i	.013705 +.035115i	.014080 +.037440i	.012905 +.038715i	.011200 +.038400i	.010105 +.036315i	.010880 +.032640i	.014905 +.027915i	.023680 +.023040i
0.3	.000105 +.042315i	.006180 +.037740i	.010105 +.036315i	.012180 +.036540i	.012825 +.037275i	.012580 +.037740i	.012105 +.037515i	.012180 +.036540i	.013705 +.035115i	.017700 +.033900i	.025305 +.033915i
0.4	.007680 +.047040i	.009705 +.041115i	.011200 +.038400i	.012105 +.037515i	.012480 +.037440i	.012505 +.037515i	.012480 +.037440i	.012825 +.037275i	.014080 +.037440i	.016905 +.038715i	.022080 +.042240i
0.5	.015625 +.046875i	.013780 +.041340i	.012905 +.038715i	.012580 +.037740i	.012505 +.037515i	.012500 +.037500i	.012505 +.037515i	.012580 +.037740i	.012905 +.038715i	.013780 +.041340i	.015625 +.046875i
0.6	.022080 +.042240i	.016905 +.038715i	.014080 +.037440i	.012825 +.037275i	.012480 +.037440i	.012505 +.037515i	.012480 +.037440i	.012105 +.037515i	.011200 +.038400i	.009705 +.041115i	.007680 +.047040i
0.7	.025305 +.033915i	.017700 +.033900i	.013705 +.035115i	.012180 +.036540i	.012105 +.037515i	.012580 +.037740i	.012825 +.037275i	.012180 +.036540i	.010105 +.036315i	.006180 +.037740i	.000105 +.042315i
0.8	.023680 +.023040i	.014905 +.027915i	.010880 +.032640i	.010105 +.036315i	.011200 +.038400i	.012905 +.038715i	.014080 +.037440i	.013705 +.035115i	.010880 +.032640i	.004825 +.031275i	−.005120 +.032640i
0.9	.015705 +.011115i	.007380 +.022140i	.004825 +.031275i	.006180 +.037740i	.009705 +.041115i	.013780 +.041340i	.016905 +.038715i	.017700 +.033900i	.014905 +.027915i	.007380 +.022140i	−.005895 +.018315i
1.0	.000000 +.000000i	−.005895 +.018315i	−.005120 +.032640i	.000105 +.042315i	.007680 +.047040i	.015625 +.046875i	.022080 +.042240i	.025305 +.033915i	.023680 +.023040i	.015705 +.011115i	.000000 +.000000i

TABLE OF COEFFICIENTS: 5-POINT

$L_{4+i}^{(4)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	.00000 +.00000i	.08550 +.00855i	.14400 +.02880i	.17850 +.05355i	.19200 +.07680i	.18750 +.09375i	.16800 +.10080i	.13650 +.09555i	.09600 +.07680i	.04950 +.04455i	.00000 +.00000i
0.1	.01350 +.08955i	.08200 +.07380i	.13130 +.07215i	.16320 +.07740i	.17950 +.08355i	.18200 +.08580i	.17250 +.08055i	.15280 +.06540i	.12470 +.03915i	.09000 +.00180i	.05050 −.04545i
0.2	.04800 +.15680i	.09770 +.12515i	.13600 +.10880i	.16350 +.10055i	.18080 +.09440i	.18850 +.08555i	.18720 +.07040i	.17750 +.04655i	.16000 +.01280i	.13530 −.03085i	.10400 −.08320i
0.3	.09450 +.20055i	.12480 +.16140i	.15150 +.13755i	.17400 +.12180i	.19170 +.10815i	.20400 +.09180i	.21030 +.06915i	.21000 +.03780i	.20250 −.00345i	.18720 −.05460i	.16350 −.11445i
0.4	.14400 +.22080i	.15550 +.18255i	.17120 +.15840i	.18930 +.14115i	.20800 +.12480i	.22550 +.10455i	.24000 +.07680i	.24970 +.03915i	.25280 −.00960i	.24750 −.06945i	.23200 −.13920i
0.5	.18750 +.21875i	.18200 +.18980i	.18850 +.17255i	.20400 +.15980i	.22550 +.14555i	.25000 +.12500i	.27450 +.09455i	.29600 +.05180i	.31150 −.00445i	.31800 −.07420i	.31250 −.15625i
0.6	.21600 +.19680i	.19650 +.18555i	.19680 +.18240i	.21270 +.18015i	.24000 +.17280i	.27450 +.15555i	.31200 +.12480i	.34830 +.07815i	.37920 +.01440i	.40050 −.06645i	.40800 −.16320i
0.7	.22050 +.15855i	.19120 +.17340i	.18950 +.19155i	.21000 +.20580i	.24730 +.21015i	.29600 +.19980i	.35070 +.17115i	.40600 +.12180i	.45650 +.05055i	.49680 −.04260i	.52150 −.15645i
0.8	.19200 +.10880i	.15830 +.15815i	.16000 +.20480i	.19050 +.24155i	.24320 +.26240i	.31150 +.26255i	.38880 +.23840i	.46850 +.18755i	.54400 +.10880i	.60870 +.00215i	.65600 −.13120i
0.9	.12150 +.05355i	.09000 +.14580i	.10170 +.22815i	.14880 +.29340i	.22350 +.33555i	.31800 +.34980i	.42450 +.33255i	.53520 +.28140i	.64230 +.19515i	.73800 +.07380i	.81450 −.08145i
1.0	.00000 +.00000i	−.02150 +.14355i	.00800 +.26880i	.07950 +.36855i	.18400 +.43680i	.31250 +.46875i	.45600 +.46080i	.60550 +.41055i	.75200 +.31680i	.88650 +.17955i	1.00000 +.00000i

TABLE OF COEFFICIENTS: 5-POINT

$L_\epsilon^{(0)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	.00000 +.00000i	.06327 +.03591i	.09792 +.06336i	.11067 +.08211i	.10752 +.09216i	.09375 +.09375i	.07392 +.08736i	.05187 +.07371i	.03072 +.05376i	.01287 +.02871i	.00000 +.00000i
0.1	−.02213 +.08271i	.04572 +.08676i	.08567 +.08931i	.10420 +.08940i	.10707 +.08631i	.09932 +.07956i	.08527 +.06891i	.06852 +.05436i	.05195 +.03615i	.03772 +.01476i	.02727 −.00909i
0.2	−.00928 +.16576i	.05595 +.13915i	.09472 +.11776i	.11327 +.09991i	.11712 +.08416i	.11107 +.06931i	.09920 +.05440i	.08487 +.03871i	.07072 +.02176i	.05867 +.00331i	.04992 −.01664i
0.3	.03687 +.24171i	.09252 +.18636i	.12387 +.14271i	.13692 +.10836i	.13695 +.08115i	.12852 +.05916i	.11547 +.04071i	.10092 +.02436i	.08727 +.00891i	.07620 −.00660i	.06867 −.02289i
0.4	.11392 +.30336i	.15327 +.22191i	.17120 +.15840i	.17347 +.10971i	.16512 +.07296i	.15047 +.04551i	.13312 +.02496i	.11595 +.00915i	.10112 −.00384i	.09007 −.01569i	.08352 −.02784i
0.5	.21875 +.34375i	.23532 +.23956i	.23407 +.15931i	.22052 +.09916i	.19947 +.05551i	.17500 +.02500i	.15047 +.00451i	.12852 −.00884i	.11107 −.01769i	.09932 −.02444i	.09375 −.03125i
0.6	.34752 +.35616i	.33507 +.23331i	.30912 +.14016i	.27495 +.07215i	.23712 +.02496i	.19947 −.00549i	.16512 −.02304i	.13647 −.03129i	.11520 −.03360i	.10227 −.03309i	.09792 −.03264i
0.7	.49567 +.33411i	.44820 +.19740i	.39227 +.09591i	.33292 +.02436i	.27447 −.02229i	.22052 −.04884i	.17395 −.05985i	.13692 −.05964i	.11087 −.05229i	.09652 −.04164i	.09387 −.03129i
0.8	.65792 +.27136i	.56967 +.12631i	.47872 +.02176i	.38987 −.04829i	.30720 −.08960i	.23407 −.10769i	.17312 −.10784i	.12627 −.09509i	.09472 −.07424i	.07895 −.04985i	.07872 −.02624i
0.9	.82827 +.16191i	.69372 +.01476i	.56295 −.08685i	.44052 −.14964i	.33027 −.18009i	.23532 −.18444i	.15807 −.16869i	.10020 −.13860i	.06267 −.09969i	.04572 −.05724i	.04887 −.01629i
1.0	1.00000 +.00000i	.81387 −.14229i	.63872 −.23424i	.47887 −.28329i	.33792 −.29664i	.21875 −.28125i	.12352 −.24384i	.05367 −.19089i	.00992 −.12864i	−.00773 −.06309i	.00000 +.00000i

TABLE OF COEFFICIENTS: 6-POINT

$L_0^{(6)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	1.00000000 +.00000000i	.80177625 +.16437375i	.61344000 +.26496000i	.44282875 +.31282125i	.29568000 +.31872000i	.17578125 +.29296875i	.08512000 +.24528000i	.02403375 +.18461625i	−.00864000 +.11904000i	−.01541375 +.05556375i	.00000000 +.00000000i
0.1	.80177625 −.16437375i	.66789000 +.00000000i	.53382875 +.10857375i	.40758000 +.17331000i	.29518125 +.20002125i	.20087000 +.19823000i	.12723375 +.17596875i	.07536000 +.14073000i	.04498625 +.09216625i	.03465000 +.05724000i	.04183875 +.01956375i
0.2	.61344000 −.26496000i	.53382875 −.10857375i	.44608000 +.00000000i	.35818125 +.07077375i	.27632000 +.10976000i	.20503375 +.12382125i	.14736000 +.11952000i	.10498625 +.10296875i	.07840000 +.07968000i	.06703875 +.05441625i	.06944000 +.03104000i
0.3	.44282875 −.31282125i	.40758000 −.17331000i	.35818125 −.07077375i	.30247000 +.00000000i	.24663375 +.04437375i	.19536000 +.06771000i	.15198625 +.07522125i	.11865000 +.07182000i	.09643875 +.06196875i	.08554000 +.04953000i	.08539125 +.03761625i
0.4	.29568000 −.31872000i	.29518125 −.20002125i	.27632000 −.10976000i	.24663375 −.04437375i	.21216000 +.00000000i	.17758625 +.02737375i	.14640000 +.04176000i	.12103875 +.04702125i	.10304000 +.04672000i	.09319125 +.04396875i	.09169000 +.04128000i
0.5	.17578125 −.29296875i	.20087000 −.19823000i	.20503375 −.12382125i	.19536000 −.06771000i	.17758625 −.02737375i	.15625000 +.00000000i	.13483875 +.01737375i	.11594000 +.02771000i	.10139125 +.03382125i	.09243000 +.03822000i	.08984375 +.04296875i
0.6	.08512000 −.24528000i	.12723375 −.17596875i	.14736000 −.11952000i	.15198625 −.07522125i	.14640000 −.04176000i	.13483875 −.01737375i	.12064000 +.00000000i	.10639125 +.01257375i	.09408000 +.02256000i	.08524375 +.03202125i	.08112000 +.04272000i
0.7	.02403375 −.18461625i	.07536000 −.14073000i	.10498625 −.10296875i	.11865000 −.07182000i	.12103875 −.04702125i	.11594000 −.02771000i	.10639125 −.01257375i	.09483000 +.00000000i	.08324375 +.01177375i	.07332000 +.02451000i	.06659625 +.03982125i
0.8	−.00864000 −.11904000i	.04498625 −.09216625i	.07840000 −.07968000i	.09643875 −.06196875i	.10304000 −.04672000i	.10139125 −.03382125i	.09408000 −.02256000i	.08324375 −.01177375i	.07072000 +.00000000i	.05819625 +.01437375i	.04736000 +.03296000i
0.9	−.01541375 −.05556375i	.03465000 −.05724000i	.06703875 −.05441625i	.08554000 −.04953000i	.09319125 −.04396875i	.09243000 −.03822000i	.08524375 −.03202125i	.07332000 −.02451000i	.05819625 −.01437375i	.04141000 +.00000000i	.02464875 +.02037375i
1.0	.00000000 +.00000000i	.04183875 −.01956375i	.06944000 −.03104000i	.08539125 −.03761625i	.09169000 −.04128000i	.08984375 −.04296875i	.08112000 −.04272000i	.06659625 −.03982125i	.04736000 −.03296000i	.02464875 −.02037375i	.00000000 +.00000000i

TABLE OF COEFFICIENTS: 6-POINT

$L_1^{(6)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	.000000 +.000000i	−.042047 −.157149i	−.016704 −.293368i	.062679 −.410907i	.182272 −.485376i	.328125 −.515025i	.486528 −.498624i	.644371 −.434343i	.789504 −.325632i	.911097 −.178101i	1.000000 +.000000i
0.1	.122940 −.078147i	.078192 −.169416i	.096795 −.256055i	.138648 −.326064i	.223041 −.370383i	.329024 −.382872i	.445767 −.360141i	.562920 −.301440i	.670973 −.208539i	.761616 −.085008i	.828099 +.000003i
0.2	.183168 −.146304i	.144911 −.196823i	.146944 −.230912i	.183037 −.267381i	.242240 −.287680i	.319243 −.285779i	.404736 −.258048i	.490769 −.203137i	.570112 −.121856i	.636615 −.017055i	.685568 +.106496i
0.3	.196707 −.197421i	.171160 −.201120i	.173913 −.213819i	.200256 −.225288i	.244639 −.227577i	.301032 −.214896i	.363285 −.183495i	.425488 −.131544i	.483331 −.059013i	.529464 +.032448i	.563857 +.139629i
0.4	.178176 −.227328i	.168909 −.206817i	.177152 −.198336i	.200355 −.192785i	.235008 −.182784i	.277001 −.163053i	.321984 −.129792i	.365727 −.080811i	.404480 −.015360i	.435333 +.055991i	.456576 +.161472i
0.5	.140625 −.234375i	.148928 −.200834i	.165271 −.180293i	.188904 −.164532i	.217907 −.148051i	.250000 −.125000i	.282003 −.092049i	.311096 −.046768i	.334729 +.012193i	.351072 +.085224i	.359375 +.171875i
0.6	.095424 −.219072i	.120667 −.183091i	.145920 −.157440i	.171873 −.135489i	.198016 −.119808i	.222999 −.097047i	.244902 −.066816i	.262045 −.026565i	.272448 +.025536i	.275091 +.090717i	.269824 +.169728i
0.7	.052143 −.183729i	.092136 −.151248i	.125669 −.129087i	.154513 −.112056i	.179115 −.095805i	.198968 −.076704i	.212961 −.051723i	.219744 −.018312i	.218087 +.025719i	.207240 +.082320i	.187293 +.153321i
0.8	.018432 −.132096i	.060785 −.110245i	.100888 −.095744i	.141231 −.084963i	.164864 −.074752i	.180757 −.062321i	.188160 −.045120i	.185963 −.020719i	.173056 +.013312i	.148689 +.059523i	.112832 +.120704i
0.9	−.000099 −.069003i	.036384 −.061992i	.102627 −.058761i	.135480 −.057360i	.158233 −.056850i	.170976 −.052728i	.172959 −.045717i	.162952 −.032736i	.139605 −.011235i	.101808 +.021816i	.049051 +.070047i
1.0	.000000 +.000000i	−.009008 −.009999i	.106496 −.019968i	.139629 −.029757i	.161472 −.038976i	.171875 −.046875i	.169728 −.052224i	.153321 −.053193i	.120704 −.047232i	.070047 −.030951i	.000000 +.000000i

TABLE OF COEFFICIENTS: 6-POINT

$L_n^{(6)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	.00000000 +.00000000i	.01352475 −.00162450i	.02339200 +.00201600i	.02896425 +.00874650i	.03014400 +.01651200i	.02734375 +.02343750i	.02145600 +.02788800i	.01382325 +.02852850i	.00620800 +.02438400i	.00076275 +.01489950i	.00000000 +.00000000i
0.1	.00589950 +.01149975i	.01365300 +.00774900i	.01968900 +.00832075i	.02329500 +.01156500i	.02423850 +.01592175i	.02273700 +.01998100i	.01942800 +.02254275i	.01533900 +.02267700i	.01185750 +.01978375i	.01070100 +.01365300i	.01388700 +.00452475i
0.2	.01238400 +.01699200i	.01558375 +.01280500i	.01849600 +.01196800i	.02036325 +.01329600i	.02084800 +.01566400i	.02000275 +.01806700i	.01824000 +.01968000i	.01630225 +.01991800i	.01523200 +.01849600i	.01634175 +.01548900i	.02118400 +.01139200i
0.3	.01803850 +.01793925i	.01787700 +.01451100i	.01841800 +.01346025i	.01887900 +.01400700i	.01884750 +.01540125i	.01824100 +.01698300i	.01727700 +.01824225i	.01644300 +.01887900i	.01646650 +.01886325i	.01828500 +.01849500i	.02301600 +.01846425i
0.4	.02188800 +.01574400i	.01954275 +.01233450i	.01848000 +.01336000i	.01794225 +.01388550i	.01747200 +.01497600i	.01688175 +.01625650i	.01622400 +.01747200i	.01576125 +.01854750i	.01593600 +.01964800i	.01734075 +.02123850i	.02068800 +.02414400i
0.5	.02343750 +.01171875i	.01998100 +.01171200i	.01806700 +.01225075i	.01696300 +.01320900i	.01625650 +.01438075i	.01562500 +.01562500i	.01500600 +.01688175i	.01446700 +.01824100i	.01419550 +.02000275i	.01446900 +.02273700i	.01562500 +.02734375i
0.6	.02251200 +.00705600i	.01892175 +.00902400i	.01688400 +.01075200i	.01570125 +.01231500i	.01497600 +.01372800i	.01438075 +.01500600i	.01372800 +.01622400i	.01292025 +.01757700i	.01192000 +.01944000i	.01071975 +.02242800i	.00931200 +.02745600i
0.7	.01924650 +.00279625i	.01636500 +.00655800i	.01479600 +.00939925i	.01400700 +.01157100i	.01358550 +.01322025i	.01320900 +.01446700i	.01261500 +.01561125i	.01157100 +.01644300i	.00984450 +.01780225i	.00717300 +.02013900i	.00323400 +.02432325i
0.8	.01401600 −.00019200i	.01253075 +.00497350i	.01196800 +.00870400i	.01196025 +.01134450i	.01216000 +.01312000i	.01225975 +.01419550i	.01195200 +.01473600i	.01089925 +.01496650i	.00870400 +.01523200i	.00487875 +.01605750i	−.00118400 +.01820900i
0.9	.00737550 −.00126225i	.00774900 +.00479700i	.00860500 +.00907875i	.00971100 +.01197300i	.01083450 +.01371975i	.01171300 +.01446900i	.01202400 +.01434075i	.01135500 +.01348500i	.00917350 +.01214175i	.00479700 +.01070100i	−.00263700 +.00976275i
1.0	.00000000 +.00000000i	.00249975 +.00363300i	.00499200 +.01081600i	.00743925 +.01378400i	.00974400 +.01831200i	.01171875 +.01562500i	.01305600 +.01468800i	.01329625 +.01251600i	.01180800 +.00918400i	.00773775 +.00488700i	.00000000 +.00000000i

TABLE OF COEFFICIENTS: 6-POINT

$L_i^{(0)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	.000000 +.000000i	.122940 +.078147i	.183168 +.146304i	.196707 +.197421i	.178176 +.227328i	.140625 +.234375i	.095424 +.219072i	.052143 +.183729i	.018432 +.132096i	−.000099 +.069003i	.000000 +.000000i
0.1	−.042047 +.157149i	.078192 +.169416i	.144911 +.186823i	.171160 +.201120i	.168909 +.206817i	.148928 +.200824i	.120667 +.182091i	.092136 +.151248i	.069785 +.110248i	.056384 +.061992i	.060003 +.009999i
0.2	−.016704 +.296368i	.086795 +.256065i	.146944 +.230912i	.173913 +.213819i	.177152 +.198336i	.165271 +.180293i	.145920 +.157440i	.125669 +.129087i	.109888 +.095744i	.102627 +.058761i	.106496 +.019968i
0.3	.063679 +.410007i	.138648 +.326064i	.182037 +.267381i	.200256 +.225288i	.200355 +.192735i	.188904 +.164832i	.171873 +.138489i	.154512 +.112056i	.141231 +.084963i	.135480 +.057360i	.139629 +.029757i
0.4	.182272 +.485376i	.223041 +.370383i	.242340 +.287680i	.244639 +.227577i	.235008 +.182784i	.217997 +.148051i	.198016 +.119808i	.179115 +.095805i	.164864 +.074752i	.158233 +.055950i	.161472 +.038976i
0.5	.328125 +.515625i	.329024 +.382872i	.319243 +.285779i	.301032 +.214896i	.277001 +.163053i	.250000 +.125000i	.222999 +.097047i	.196968 +.076704i	.180757 +.062321i	.170976 +.052728i	.171875 +.046875i
0.6	.486528 +.498624i	.445767 +.360141i	.404736 +.258048i	.363285 +.183495i	.321984 +.129792i	.282003 +.092049i	.244992 +.066816i	.212961 +.051723i	.188160 +.045120i	.172959 +.045717i	.169728 +.052224i
0.7	.644371 +.434343i	.562920 +.301440i	.490769 +.203137i	.425488 +.131544i	.365727 +.080811i	.311096 +.046768i	.262045 +.026565i	.219744 +.018312i	.185963 +.020719i	.162952 +.032736i	.153321 +.053193i
0.8	.789504 +.325632i	.670973 +.208539i	.570112 +.121856i	.482331 +.059013i	.404480 +.015360i	.334729 −.012193i	.272448 −.025536i	.218067 −.025719i	.173056 −.013312i	.139605 +.011235i	.120704 +.047232i
0.9	.911097 +.178101i	.761616 +.085608i	.636615 +.017055i	.529464 −.032448i	.435333 −.059991i	.351072 −.085224i	.275091 −.090717i	.207240 −.082320i	.148689 −.059523i	.101808 −.021816i	.070047 +.030951i
1.0	1.000000 +.000000i	.828099 −.000903i	.685568 −.106496i	.563857 −.139629i	.456576 −.161472i	.359375 −.171875i	.269824 −.169728i	.187293 −.153321i	.112832 −.120704i	.049051 −.070047i	.000000 +.000000i

TABLE OF COEFFICIENTS: 6-POINT

$L_{1+i}^{(6)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	.0000000 +.0000000i	.0078075 −.0722475i	.1843200 −.0979200i	.2507925 −.0001425i	.2918400 −.0614400i	.3046875 −.0234375i	.2889600 +.0134400i	.2463825 +.0426675i	.1804800 +.0499200i	.0962775 +.0373725i	.0000000 +.0000000i
0.1	.0078075 +.0722475i	.1484200 +.0000000i	.1991925 −.0358475i	.2414400 −.0454200i	.2690875 −.0385425i	.2784600 −.0244400i	.2679825 −.0114375i	.2378800 −.0066600i	.1898775 −.0157325i	.1269000 −.0424800i	.0527725 −.0886275i
0.2	.1843200 +.0979200i	.1991925 +.0358475i	.2230400 +.0000000i	.2470875 −.0170475i	.2649600 −.0227200i	.2723825 −.0241425i	.2668800 −.0278400i	.2474775 −.0394375i	.2144000 −.0633600i	.1687725 −.1025325i	.1123200 −.1580800i
0.3	.2507925 +.0901425i	.2414400 +.0454200i	.2470875 +.0170475i	.2592600 +.0000000i	.2715825 −.0110475i	.2794800 −.0214200i	.2798775 −.0361425i	.2706000 −.0596400i	.2515725 −.0954375i	.2215200 −.1458900i	.1806675 −.2117325i
0.4	.2918400 +.0614400i	.2690875 +.0385425i	.2649600 +.0227200i	.2715825 +.0110475i	.2828800 +.0000000i	.2942775 −.0142475i	.3024000 −.0355200i	.3047725 −.0673425i	.2995200 −.1126400i	.2850675 −.1734375i	.2598400 −.2505600i
0.5	.3046875 +.0234375i	.2784600 +.0244400i	.2723825 +.0241425i	.2794800 +.0214200i	.2942775 +.0142475i	.3125000 +.0000000i	.3307725 −.0243475i	.3463200 −.0614200i	.3566675 −.1141425i	.3593400 −.1844400i	.3515625 −.2734375i
0.6	.2889600 −.0134400i	.2679825 +.0114375i	.2668800 +.0278400i	.2798775 +.0361425i	.3024000 +.0355200i	.3307725 +.0243475i	.3619200 +.0000000i	.3930675 −.0398475i	.4214400 −.0979200i	.4439625 −.1765425i	.4569600 −.2774400i
0.7	.2463825 −.0426675i	.2378800 +.0066600i	.2474775 +.0394375i	.2706000 +.0596400i	.3047725 +.0673425i	.3463200 +.0614200i	.3930675 +.0398475i	.4425400 +.0000000i	.4919625 −.0610475i	.5379600 −.1462200i	.5762575 −.2581425i
0.8	.1804800 −.0499200i	.1898775 +.0157325i	.2144000 +.0633600i	.2515725 +.0954375i	.2995200 +.1126400i	.3566675 +.1141425i	.4214400 +.0979200i	.4919625 +.0610475i	.5657600 +.0000000i	.6394575 −.0890475i	.7084800 −.2099200i
0.9	.0962775 −.0373725i	.1269000 +.0424800i	.1687725 +.1025325i	.2215200 +.1458900i	.2850675 +.1734375i	.3593400 +.1844400i	.4439625 +.1765425i	.5379600 +.1462200i	.6394575 +.0890475i	.7453800 +.0000000i	.8511525 −.1262475i
1.0	.0000000 +.0000000i	.0527725 +.0886275i	.1123200 +.1580800i	.1806675 +.2117325i	.2598400 +.2505600i	.3515625 +.2734375i	.4569600 +.2774400i	.5762575 +.2581425i	.7084800 +.2099200i	.8511525 +.1262475i	1.0000000 +.0000000i

TABLE OF COEFFICIENTS: 6-POINT

$L_{2i}^{(6)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	.00000000 +.00000000i	.00689050 −.01149975i	.01238400 −.01692200i	.01802850 −.01793925i	.02188800 −.01574400i	.02343750 −.01171875i	.02251200 −.00705600i	.01924650 −.00279825i	.01401600 +.00019200i	.00737550 +.00126225i	.00000000 +.00000000i
0.1	.01352475 +.00162450i	.01365300 −.00774900i	.01558375 −.01280500i	.01787700 −.01451100i	.01954275 −.01353450i	.01968100 −.01171300i	.01892175 −.00902400i	.01636500 −.00655500i	.01253075 −.00497350i	.00774900 −.00479700i	.00349075 −.00363600i
0.2	.02339200 −.00201600i	.01968900 −.00632075i	.01849600 −.01196800i	.01841800 −.01346025i	.01848000 −.01336000i	.01806700 −.01225975i	.01666400 −.01075200i	.01479600 −.00839925i	.01196800 −.00870400i	.00860500 −.00907875i	.00499200 −.01081600i
0.3	.02896425 −.00674650i	.02329500 −.01156500i	.02036325 −.01329600i	.01887900 −.01407700i	.01794225 −.01385550i	.01698300 −.01320000i	.01570125 −.01231500i	.01400700 −.01157100i	.01196025 −.01134450i	.00971100 −.01197300i	.00743925 −.01374600i
0.4	.03014400 −.01651200i	.02423850 −.01521750i	.02084800 −.01566400i	.01884750 −.01561125i	.01747200 −.01497600i	.01625650 −.01438875i	.01497600 −.01372800i	.01358550 −.01322025i	.01216000 −.01312000i	.01083450 −.01371975i	.00974400 −.01531200i
0.5	.02734375 −.02343750i	.02273700 −.01998100i	.02000275 −.01806700i	.01834100 −.01693000i	.01688175 −.01625650i	.01562500 −.01562500i	.01438075 −.01500600i	.01320900 −.01446700i	.01225975 −.01419550i	.01171300 −.01448000i	.01171875 −.01562500i
0.6	.02145600 −.02788800i	.01942800 −.02254275i	.01824000 −.01968000i	.01727700 −.01824225i	.01622400 −.01747200i	.01500600 −.01688175i	.01372800 −.01623400i	.01261500 −.01546125i	.01195200 −.01473600i	.01202400 −.01434075i	.01305600 −.01468800i
0.7	.01382325 −.02852850i	.01533900 −.02267700i	.01630225 −.01991800i	.01644300 −.01887900i	.01576125 −.01854750i	.01446700 −.01824100i	.01292025 −.01757700i	.01157100 −.01643000i	.01089925 −.01496650i	.01135500 −.01348500i	.01320825 −.01251600i
0.8	.00620800 −.02438400i	.01185750 −.01978375i	.01523200 −.01849600i	.01646650 −.01886325i	.01593600 −.01964800i	.01419550 −.02002775i	.01192000 −.01944000i	.00984450 −.01780225i	.00870400 −.01323200i	.00917350 −.01214175i	.01180800 −.00918400i
0.9	.00076275 −.01489050i	.01070100 −.01365300i	.01634175 −.01548900i	.01828500 −.01846500i	.01734075 −.02123850i	.01446900 −.02273700i	.01071975 −.02342850i	.00717300 −.02013900i	.00487875 −.01605750i	.00479700 −.01070100i	.00773775 −.00488700i
1.0	.00000000 +.00000000i	.01388700 −.00452475i	.02118400 −.01139200i	.02301600 −.01846425i	.02068800 −.02414400i	.01562500 −.02734375i	.00931200 −.02745600i	.00323400 −.02432325i	−.00118400 −.01820600i	−.00263700 −.00976275i	.00000000 +.00000000i

TABLE OF COEFFICIENTS: 7-POINT

$L_0^{(0)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	1.000000000 +.000000000i	.766417725 +.173834325i	.553766400 +.268300800i	.370920025 +.301852425i	.222873600 +.291379200i	.111328125 +.251953125i	.035257600 +.196627200i	−.008541975 +.136288425i	−.024921600 +.079564800i	−.019866375 +.032786325i	.000000000 +.000000000i
0.1	.779165775 −.193157325i	.627816600 −.013357800i	.480445875 +.098616375i	.347052600 +.157192200i	.234047775 +.175824675i	.144785400 +.166663800i	.080075475 +.140395575i	.038679000 +.100047000i	.017784975 +.070881075i	.013469400 +.040339800i	.021136275 +.018041175i
0.2	.567705600 −.303438800i	.484548025 −.132637575i	.392550400 −.017843200i	.302287725 +.052286325i	.221056000 +.087808000i	.153349225 +.098204825i	.101318400 +.091948800i	.065212525 +.076317925i	.043801600 +.057267200i	.034781625 +.039353625i	.035161000 +.025715200i
0.3	.378720475 −.347191425i	.349491000 −.196737000i	.302373975 −.089519925i	.248025400 −.018146200i	.194149275 +.024746175i	.145020600 +.046198200i	.106390375 +.053654875i	.076872600 +.049774200i	.057313275 +.042278175i	.046641400 +.033859800i	.043101975 +.027144075i
0.4	.221030400 −.340531200i	.231756525 −.217344075i	.218037600 −.126356800i	.192869625 −.060162375i	.161241600 −.016972800i	.129504525 +.009053925i	.101222400 +.022540800i	.078405225 +.027672825i	.061824000 +.028032000i	.051307725 +.026486325i	.046022400 +.025132800i
0.5	.099609375 −.298828125i	.137068600 −.206625800i	.148316275 −.134338825i	.142901400 −.080164200i	.128128975 −.041566925i	.109375000 −.015625000i	.090383475 +.000679575i	.073545400 +.010233800i	.060159775 +.015590675i	.050676600 +.018790200i	.044921875 +.021484375i
0.6	.016038400 −.236278200i	.068163225 −.175805175i	.093984000 −.125088000i	.101969725 −.084535675i	.098726400 −.053491200i	.089258025 −.030691575i	.077209600 −.014476800i	.065092125 −.003094875i	.054489600 +.005107200i	.046248025 +.011536425i	.040646400 +.017260800i
0.7	−.031023525 −.165499425i	.025305400 −.134992200i	.057176775 −.105512325i	.072000600 −.079249800i	.075322875 −.057122625i	.071532600 −.039157800i	.063950775 −.024818325i	.055001400 −.013276200i	.046364475 −.005314425i	.039111000 +.004923000i	.033819975 +.012990075i
0.8	−.045350400 −.097238800i	.009224125 −.092868875i	.037273600 −.082508800i	.053324025 −.069691575i	.058854400 −.056499200i	.057449725 −.043951675i	.051936000 −.032352000i	.044497225 −.021577175i	.036774400 −.011318200i	.029946525 −.001948075i	.024793600 +.008819200i
0.9	−.032642225 −.040013325i	.007565400 −.056428200i	.032195475 −.061720425i	.045019000 −.060333000i	.049194975 −.055112925i	.047369400 −.047728200i	.041756275 −.039030825i	.034200600 −.029365800i	.026233375 −.018826125i	.019048600 −.007453800i	.013612275 +.004613175i
1.0	.000000000 +.000000000i	.024363225 −.030767175i	.038822400 −.047347200i	.045276525 −.054612075i	.045465600 −.055756800i	.041015625 −.052734375i	.033465600 −.046636800i	.024276525 −.038022075i	.014822400 −.027187200i	.006363225 −.014387175i	.000000000 +.000000000i

TABLE OF COEFFICIENTS: 7-POINT

$L_1^{(0)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	.0000000 +.0000000i	−.1316852 −.2089449i	−.1427328 −.4110336i	−.0502008 −.5766621i	.0913408 −.6856704i	.2812500 −.7265625i	.4841088 −.6956544i	.6758752 −.5961501i	.8358912 −.4371456i	.9477468 −.2325609i	1.0000000 +.0000000i
0.1	.1352952 −.1809351i	.0347608 −.2762784i	.0019440 −.3847455i	.0498168 −.4793424i	.1491672 −.5410431i	.2799672 −.5581536i	.4225968 −.5256039i	.5589240 −.4441680i	.6732408 −.3196119i	.7530552 −.1617096i	.7897392 +.0164529i
0.2	.1686528 −.3147264i	.0928772 −.3319121i	.0755712 −.3736576i	.1072248 −.4161429i	.1756160 −.4421120i	.2671068 −.4403800i	.3677952 −.4052736i	.4645232 −.3359861i	.5457408 −.2358784i	.6022260 −.1116945i	.6276608 +.0272896i
0.3	.1373308 −.3943779i	.1018360 −.3641520i	.0997908 −.3629259i	.1276632 −.3704736i	.1789252 −.3718011i	.2451768 −.3567984i	.3172260 −.3198195i	.3860248 −.2591904i	.4434612 −.1766451i	.4830072 −.0766896i	.5002228 +.0341061i
0.4	.0724992 −.4202496i	.0767052 −.3683121i	.0935808 −.3442944i	.1244040 −.3318405i	.1671168 −.3185664i	.2174772 −.2958561i	.2700672 −.2585856i	.3191568 −.2047749i	.3594240 −.1351680i	.3865308 −.0527409i	.3975552 +.0378624i
0.5	.0000000 −.3984375i	.0381368 −.3452384i	.0728712 −.3147631i	.1088952 −.2946576i	.1474688 −.2753519i	.1875000 −.2500000i	.2264808 −.2143479i	.2612792 −.1665296i	.2887872 −.1067911i	.3064248 −.0371424i	.3125000 +.0390625i
0.6	−.0007488 −.3392256i	.0021988 −.2999049i	.0503040 −.2753280i	.0904488 −.2571309i	.1261312 −.2386176i	.1584732 −.2148441i	.1870848 −.1826304i	.2107840 −.1405005i	.2281728 −.0885504i	.2380692 −.0282441i	.2397952 +.0378624i
0.7	−.0963062 −.2556099i	−.0196488 −.2400816i	.0351372 −.2297931i	.0760728 −.2202144i	.1078860 −.2074395i	.1330488 −.1884144i	.1525692 −.1610931i	.1666392 −.1245216i	.1751388 −.0788499i	.1779960 −.0252720i	.1754028 +.0341061i
0.8	−.0986112 −.1618944i	−.0204340 −.1750745i	.0332928 −.1836544i	.0704472 −.1866381i	.0960512 −.1826816i	.1131348 −.1704649i	.1234560 −.1489920i	.1280768 −.1178189i	.1277952 −.0772096i	.1234332 −.0282201i	.1159808 +.0272896i
0.9	−.0656568 −.0723591i	.0025912 −.1145376i	.0475488 −.1430550i	.0760440 −.1600080i	.0925288 −.1662399i	.0998712 −.1618656i	.1000032 −.1467111i	.0944248 −.1206864i	.0845640 −.0840855i	.0719928 −.0378144i	.0584992 +.0164529i
1.0	.0000000 +.0000000i	.0483588 −.0673569i	.0778752 −.1138176i	.0933912 −.1439781i	.0979968 −.1603584i	.0937500 −.1640625i	.0823528 −.1553664i	.0650632 −.1342341i	.0440832 −.1007616i	.0215028 −.0555489i	.0000000 +.0000000i

TABLE OF COEFFICIENTS: 7-POINT

$L_2^{(6)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	.000000000 +.000000000i	.010438200 −.027321525i	.027020800 −.040089600i	.043833300 −.040492725i	.056563200 −.031718400i	.062500000 −.017578125i	.060499200 −.002150400i	.050910300 +.010558275i	.035468800 +.016934400i	.017152200 +.014060475i	.000000000 +.000000000i
0.1	.025309050 −.001492225i	.027010800 −.018966600i	.032697450 −.027951525i	.040526000 −.029193000i	.047289450 −.024452025i	.050434800 −.016122600i	.049045050 −.009107025i	.043285200 +.000468600i	.034412250 +.003576375i	.024649200 +.000516600i	.017023050 −.000814725i
0.2	.043891200 −.011174400i	.036796500 −.019365125i	.034339200 −.023718400i	.038893800 −.023980725i	.041740800 −.020825600i	.043102700 −.015550525i	.042144000 −.009792000i	.038038200 −.005258525i	.034380800 −.003481600i	.030111300 −.005584725i	.028339200 −.013070400i
0.3	.048498450 −.023499525i	.040084800 −.023808600i	.037121050 −.023730225i	.037027200 −.022394400i	.037835250 −.019375125i	.038243200 −.015473400i	.037633050 −.011418225i	.036052800 −.008160600i	.034162450 −.006555525i	.033144000 −.007167000i	.034578450 −.010091025i
0.4	.044620800 −.034329600i	.038011200 −.026830525i	.035136000 −.025248000i	.034370700 −.022170525i	.034444800 −.018999600i	.034513800 −.015568725i	.034195200 −.012230400i	.033568500 −.009361125i	.033139200 −.007334400i	.033766800 −.006331725i	.036556800 −.006301600i
0.5	.035156250 −.041015625i	.032245200 −.032107400i	.031101050 −.026390725i	.030946800 −.022266600i	.031137450 −.018820025i	.031250000 −.015625000i	.031137450 −.012567525i	.030946800 −.009866600i	.031101050 −.007033225i	.032245200 −.004547400i	.035156250 −.001953125i
0.6	.023116800 −.042201600i	.024714300 −.032417325i	.026099200 −.026054400i	.027316000 −.021766125i	.027955200 −.018470400i	.028261300 −.015568725i	.028204800 −.012729600i	.028018300 −.009765525i	.028006000 −.006528000i	.028958700 −.002820525i	.031180800 +.001670400i
0.7	.011370450 −.037653525i	.017364000 −.029127000i	.021357450 −.023813025i	.023873800 −.020340600i	.025228050 −.017770725i	.025663200 −.015473400i	.025430250 −.013022625i	.024547200 −.010109400i	.024316050 −.006472725i	.024304800 −.001848600i	.025263450 +.004029975i
0.8	.002419200 −.029070400i	.011953800 −.022794725i	.018060800 −.019801600i	.021677700 −.018063525i	.023424000 −.016832000i	.023745200 −.015550525i	.023020800 −.013785600i	.021634300 −.011175725i	.020019200 −.007394400i	.018639000 −.002155125i	.017971200 +.004826400i
0.9	−.001786050 −.014877225i	.009889200 −.014243400i	.017202250 −.014581125i	.021325200 −.015114000i	.023035050 −.015963225i	.022874800 −.016122600i	.021279450 −.015399525i	.018666000 −.013413000i	.015467450 −.009794025i	.012250800 −.004266600i	.009499050 +.003613275i
1.0	.000000000 +.000000000i	.012090700 −.004749525i	.019468800 −.009956600i	.023347800 −.012646725i	.024409200 −.015904400i	.023437500 −.017578125i	.020563200 −.018278400i	.016270800 −.017287725i	.011020800 −.014169600i	.006375700 −.008511525i	.000000000 +.000000000i

TABLE OF COEFFICIENTS: 7-POINT

$L_i^{(6)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	.00000000 +.00000000i	.07772805 +.13571415i	.09169920 +.22325760i	.06849045 +.26816135i	.03887680 +.27095040i	−.01171875 +.24609375i	−.04273920 +.20106240i	−.05797155 +.14549535i	−.05498880 +.08847360i	−.03455595 +.03790215i	.00000000 +.00000000i
0.1	−.11276405 +.13822785i	−.00195480 +.19613160i	.04634955 +.23335065i	.05496200 +.24797400i	.04205955 +.24146265i	.02132520 +.21763560i	.00252595 +.18176385i	−.00817320 +.13977240i	−.00773925 +.09755025i	.00421480 +.06036840i	.02595195 +.03240555i
0.2	−.13605120 +.29166640i	−.01907075 +.27797975i	.03968480 +.26659840i	.06329845 +.25131135i	.06238720 +.22952960i	.05183605 +.20132815i	.03916800 +.16857600i	.03005005 +.13417415i	.02763520 +.10140160i	.03294045 +.07336935i	.04530080 +.05258240i
0.3	−.09113845 +.43284465i	.01759320 +.38628760i	.07034995 +.30435585i	.09136680 +.26158440i	.09263675 +.23431225i	.08398650 +.18974040i	.07183995 +.15709785i	.06121320 +.12091560i	.05500155 +.10040865i	.05443800 +.07896600i	.05939955 +.06374865i
0.4	.03665920 +.54005760i	.10077405 +.41877615i	.13171200 +.33158400i	.13967005 +.26832315i	.13317120 +.21672960i	.11908245 +.17643735i	.10266880 +.14327040i	.08768325 +.11600775i	.07649280 +.09431040i	.07024045 +.07824735i	.06904320 +.06792960i
0.5	.19921875 +.59765625i	.21685480 +.44984440i	.21587305 +.33701185i	.20215320 +.25791960i	.18063755 +.19960265i	.15625000 +.15625000i	.13183755 +.12368265i	.11015320 +.09859960i	.09387395 +.08258185i	.08085480 +.07174440i	.07421875 +.06640625i
0.6	.39680320 +.59502960i	.35145045 +.43128735i	.31265280 +.31319040i	.27206325 +.22862775i	.23162880 +.16823040i	.19309245 +.12543735i	.15813120 +.09576960i	.12808005 +.07621215i	.10387200 +.06470400i	.08508405 +.05973615i	.07441920 +.06005760i
0.7	.57921955 +.53009865i	.48955800 +.37080600i	.41122155 +.25643865i	.34193320 +.17853560i	.28045095 +.11950785i	.22630680 +.08174040i	.17944675 +.05790225i	.14006880 +.04486440i	.10846995 +.04032585i	.08471320 +.04244760i	.06868155 +.04959465i
0.8	.75694080 +.40458240i	.61657045 +.26520635i	.50091520 +.16668160i	.40408005 +.09839415i	.32204800 +.05273600i	.25226605 +.02432815i	.19326720 +.00936960i	.14432845 +.00509135i	.10516480 +.00931840i	.07565925 +.02013975i	.05562880 +.03568640i
0.9	.90219195 +.22365585i	.71925480 +.11940840i	.57210075 +.04718025i	.45166680 −.00110760i	.35156595 −.03102615i	.26756520 −.04636440i	.19709955 −.04974735i	.13882200 −.04314600i	.09218955 −.02827935i	.05706520 −.00690840i	.03347505 +.01897785i
1.0	1.00000000 +.00000000i	.78669405 −.05785785i	.61701120 −.09584640i	.47927845 −.11868465i	.36526080 −.12917760i	.26953125 −.12890625i	.18887680 −.11880960i	.12174045 −.09965865i	.06769920 −.07242240i	.02697805 −.03852585i	.00000000 +.00000000i

TABLE OF COEFFICIENTS: 7-POINT

$L_{1+i}^{(6)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	.00000000 +.00000000i	.25835275 −.03937275i	.42969600 +.00806400i	.51648975 +.09755025i	.52838400 +.19353600i	.48046875 +.26953125i	.39110400 +.30777600i	.28002975 +.29873025i	.16665600 +.24038400i	.06853275 +.13738725i	.00000000 +.00000000i
0.1	.13077225 +.23260275i	.28199800 +.13357800i	.39080025 +.11424775i	.45132600 +.14008200i	.46522825 +.18051075i	.43968600 +.21395400i	.38546925 +.22517175i	.31523800 +.20643400i	.24201225 +.15201075i	.17782200 +.06748200i	.13253725 −.04113225i
0.2	.29030400 +.34320600i	.34978775 +.22746425i	.40147200 +.17843200i	.43368675 +.16868925i	.44211200 +.17561600i	.42788775 +.18169225i	.39590400 +.17452800i	.35327075 +.14671325i	.30796800 +.09548800i	.26767575 +.02232225i	.23878400 −.06822400i
0.3	.43848525 +.35583975i	.42694200 +.25530600i	.43282425 +.20364675i	.44074200 +.18148200i	.44226525 +.17243175i	.43421400 +.16350600i	.41712825 +.14531475i	.39391800 +.11209800i	.36869325 +.06157575i	.34577400 −.00538200i	.32888025 −.08526525i
0.4	.54681600 +.29798400i	.48814075 +.23468325i	.46329600 +.19987200i	.45506175 +.18173025i	.45260800 +.16972800i	.44996475 +.15519525i	.44467200 +.13171200i	.43660975 +.09531825i	.42700800 +.04454400i	.41763675 −.01974075i	.41017600 −.09465600i
0.5	.59765625 +.19921875i	.51685400 +.18566600i	.47821725 +.17964775i	.46440600 +.17615400i	.46372025 +.16993475i	.46875000 +.15625000i	.47520525 +.13143975i	.48092600 +.09331400i	.48507225 +.04136275i	.48749400 −.02321400i	.48828125 −.09765625i
0.6	.58329600 +.08870400i	.50459175 +.12892425i	.46924800 +.15686400i	.46133475 +.17339325i	.46963200 +.17779200i	.48645975 +.16868025i	.50068800 +.14476800i	.53092675 +.10542325i	.54489600 +.05107200i	.55897575 −.01661175i	.56793600 −.09465600i
0.7	.50484525 −.00662025i	.44997400 +.08401800i	.43362825 +.14523075i	.44263800 +.18265800i	.46743325 +.19917975i	.50105400 +.19602600i	.53834025 +.17370675i	.57530200 +.13276200i	.60866925 +.07433175i	.63562200 +.00054600i	.65370025 −.08526525i
0.8	.37094400 −.06374400i	.35762075 +.06786725i	.37324800 +.15692800i	.40858575 +.21255825i	.45670400 +.24012800i	.51217275 +.24254725i	.57043200 +.22137600i	.62734175 +.17775425i	.67891200 +.11315200i	.72121275 +.02993925i	.75046400 −.06822400i
0.9	.19629225 −.06511725i	.23686200 +.09340200i	.29353725 +.20143575i	.36199800 +.27011400i	.43876425 +.30600675i	.52056600 +.31259400i	.60389325 +.29155575i	.68472600 +.24388200i	.75844425 +.17080275i	.81991800 +.07453800i	.86377725 −.04113225i
1.0	.00000000 +.00000000i	.10026775 +.16839225i	.20217600 +.28454400i	.30713475 +.35994525i	.41574400 +.40089600i	.52734375 +.41015625i	.63974400 +.38841600i	.74913475 +.33558525i	.85017600 +.25190400i	.93626775 +.13887225i	1.00000000 +.00000000i

TABLE OF COEFFICIENTS: 7-POINT

$L_{2+i}^{(3)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	.00000000 +.00000000i	.00657495 −.03207105i	.02119680 −.04907520i	.03753855 −.05313945i	.05099520 −.04730880i	.05850375 −.03515625i	.05886720 −.02042880i	.05160255 −.00672945i	.03809280 +.00276480i	.02000295 +.00554895i	.00000000 +.00000000i
0.1	.03207105 −.00657495i	.02671560 −.02671560i	.03011905 −.03833235i	.03710040 −.04197720i	.04381065 −.03928815i	.04767880 −.03224520i	.04732185 −.02349735i	.04241880 −.01547160i	.03354865 −.01042695i	.02199240 −.01018440i	.00949905 −.01595295i
0.2	.04907520 −.02119680i	.03832335 −.03011905i	.03568640 −.03568640i	.03714735 −.03731985i	.03906720 −.03543040i	.04116695 −.03110105i	.04047360 −.02580480i	.03723015 −.02115665i	.03176960 −.01871360i	.02495295 −.01978065i	.01797120 −.02529280i
0.3	.05313945 −.03753855i	.04197720 −.03710040i	.03731985 −.03714735i	.03629640 −.03629640i	.03609585 −.03413655i	.03698520 −.03094680i	.03629145 −.02745015i	.03434760 −.02460360i	.03140265 −.02342415i	.02809500 −.02485080i	.02529345 −.02964255i
0.4	.04730880 −.05099520i	.03928815 −.04381065i	.03543040 −.03906720i	.03413655 −.03609585i	.03394560 −.03394560i	.03388855 −.03113745i	.03344640 −.02845440i	.03253215 −.02636745i	.03143680 −.02549760i	.03073935 −.02651385i	.03118080 −.03006720i
0.5	.03515625 −.05850375i	.03224520 −.04767880i	.03110105 −.04116695i	.03094680 −.03698520i	.03113745 −.03388855i	.03125000 −.03125000i	.03113745 −.02888655i	.03094680 −.02692120i	.03110105 −.02568095i	.03224520 −.02563080i	.03515625 −.02734375i
0.6	.02042880 −.05886720i	.02349735 −.04732185i	.02580480 −.04047360i	.02745015 −.03629145i	.02845440 −.03344640i	.02888655 −.03113745i	.02895360 −.02895360i	.02905455 −.02677185i	.02979840 −.02469120i	.03198615 −.02302665i	.03655680 −.02219520i
0.7	.00672945 −.05160255i	.01547160 −.04241880i	.02115665 −.03723015i	.02460360 −.03434760i	.02636745 −.03253215i	.02692120 −.03094680i	.02677185 −.02905455i	.02655240 −.02655240i	.02707585 −.02334135i	.02935320 −.01953240i	.03457845 −.01548855i
0.8	−.00276480 −.03809280i	.01042695 −.03354865i	.01871360 −.03176960i	.02342415 −.03140265i	.02549760 −.03143680i	.02568095 −.03110105i	.02469120 −.02979840i	.02334135 −.02707585i	.02263040 −.02263040i	.02379735 −.01635105i	.02833920 −.00839680i
0.9	−.00554895 −.02000295i	.01018440 −.02199240i	.01978065 −.02495295i	.02485080 −.02809500i	.02651385 −.03073935i	.02563080 −.03224520i	.02302665 −.03198615i	.01953240 −.02935320i	.01635105 −.02379735i	.01490760 −.01490760i	.01702305 −.00252495i
1.0	.00000000 +.00000000i	.01595295 −.00949905i	.02529280 −.01797120i	.02964255 −.02529345i	.03006720 −.03118080i	.02734375 −.03515625i	.02219520 −.03655680i	.01548855 −.03457845i	.00839680 −.02833920i	.00252495 −.01702305i	.00000000 +.00000000i

TABLE OF COEFFICIENTS: 7-POINT

$L_{2i}^{(0)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	.000000000 +.000000000i	.012273525 −.001838250i	.019353600 +.000576000i	.021928725 +.004730250i	.020966400 +.008832000i	.017578125 +.011718750i	.012903400 +.012768000i	.008005725 +.011807250i	.003801600 +.009024000i	.000987525 +.004878750i	.000000000 +.000000000i
0.1	.007150725 +.011286000i	.013653000 +.005608800i	.017634825 +.004305600i	.019096200 +.005258400i	.018397125 +.009252000i	.016122600 +.008268000i	.012965625 +.006680800i	.009628200 +.007917600i	.006740325 +.006020400i	.004797000 +.003247200i	.004113225 +.000000000i
0.2	.016422400 +.015552000i	.017637925 +.008789850i	.018496000 +.005875200i	.018461125 +.005156550i	.017420800 +.005414400i	.015550525 +.005807250i	.013196900 +.005817600i	.010778125 +.005197950i	.008704000 +.003916800i	.007311925 +.002104650i	.006822400 +.000000000i
0.3	.035064025 +.013922000i	.022075800 +.008304400i	.020320125 +.005320800i	.018879000 +.004141200i	.017302425 +.003822600i	.015473400 +.003774000i	.013490625 +.003620400i	.011571000 +.003166800i	.009965625 +.002362200i	.008809800 +.001263600i	.008526525 +.000000000i
0.4	.081065600 +.008064000i	.025384125 +.004837950i	.021907200 +.003110400i	.019487325 +.002306850i	.017472000 +.001996800i	.015568725 +.001875750i	.013728000 +.001747200i	.012044325 +.001504650i	.010675200 +.001113600i	.009778125 +.000593550i	.009465600 +.000000000i
0.5	.033308125 +.000000000i	.026595400 +.000000000i	.022519225 +.000000000i	.019750600 +.000000000i	.017569525 +.000000000i	.015625000 +.000000000i	.013818025 +.000000000i	.012203600 +.000000000i	.010904725 +.000000000i	.010059400 +.000000000i	.009765625 +.000000000i
0.6	.081065600 −.008064000i	.025384125 −.004837950i	.021907200 −.003110400i	.019487325 −.002306850i	.017472000 −.001996800i	.015568725 −.001875750i	.013728000 −.001747200i	.012044325 −.001504650i	.010675200 −.001113600i	.009778125 −.000593550i	.009465600 +.000000000i
0.7	.035064025 −.013922000i	.022075800 −.008304400i	.020320125 −.005320800i	.018879000 −.004141200i	.017302425 −.003822600i	.015473400 −.003774000i	.013490625 −.003620400i	.011571000 −.003166800i	.009965625 −.002362200i	.008809800 −.001263600i	.008526525 +.000000000i
0.8	.016422400 −.015552000i	.017637925 −.008789850i	.018496000 −.005875200i	.018461125 −.005156550i	.017420800 −.005414400i	.015550525 −.005807250i	.013196900 −.005817600i	.010778125 −.005197950i	.008704000 −.003916800i	.007311925 −.002104650i	.006822400 +.000000000i
0.9	.007150725 −.011286000i	.013653000 −.005608800i	.017634825 −.004305600i	.019096200 −.005258400i	.018397125 −.009252000i	.016122600 −.008268000i	.012965625 −.006680800i	.009628200 −.007917600i	.006740325 −.006020400i	.004797000 −.003247200i	.004113225 +.000000000i
1.0	.000000000 +.000000000i	.012273525 +.001838250i	.019353600 −.000576000i	.021928725 −.004730250i	.020966400 −.008832000i	.017578125 −.011718750i	.012903400 −.012768000i	.008005725 −.011807250i	.003801600 −.009024000i	.000987525 −.004878750i	.000000000 +.000000000i

BESSEL FUNCTION EXPANSIONS OF CERTAIN COULOMB WAVE FUNCTIONS

By Milton Abramowitz

The solutions of the differential equation

$$\frac{d^2y}{d\rho^2} + \left\{1 - \frac{2\eta}{\rho} - \frac{L(L+1)}{\rho^2}\right\} y = 0 \tag{1}$$

are of importance in the problem of the motion of a particle in a Coulomb field of force. The Computation Laboratory of the National Bureau of Standards is now engaged in the tabulation of the regular solution $F_L = C_L \rho^{L+1} \phi_L(\rho, \eta)$ for $L = 0$ in the range $\eta = 0$ to $\eta = 10$ and $\rho = 0$ to $\rho = 10$ where

$$\phi_L(\rho, \eta) = \sum_{n=L+1}^{\infty} A_n^{(L)} \rho^{n-L-1} \tag{2}$$

$$A_{L+1}^{(L)} = 1; \qquad A_{L+2}^{(L)} = \eta/(L+1) \tag{3}$$

$$(n - L - 1)(n + L)A_n^{(L)} = 2\eta A_{n-1}^{(L)} - A_{n-2}^{(L)}$$

and C_L is a factor depending on η chosen so that as

$$\rho \to \infty, \; F_L \to \sin\,(\rho - \eta \ln 2\rho - \tfrac{1}{2}\pi L + \sigma_L)$$

where $\sigma_L = \arg \Gamma\,(L + 1 + i\eta)$. For large values of ρ and η the power series converges too slowly for computational purposes making it desirable to obtain a faster converging series. It is the purpose of this article to obtain an expansion for the case $L = 0$ in terms of Bessel functions of integral order. When $L = 0$ the differential equation becomes

$$y'' + (1 - 2\eta\rho^{-1})y = 0 \tag{4}$$

If we assume that the regular solution may be expressed in the form

$$y = \textstyle\sum_{n=1}^{\infty} a_n(\eta) J_n(\rho) \tag{5}$$

and substitute in the differential equation we obtain

$$\textstyle\sum_{n=1}^{\infty} a_n \{J_n'' + J_n - 2\eta\rho^{-1} J_n\} = 0. \tag{6}$$

If we now make use of the relations:

$$4J_n'' = J_{n-2} - 2J_n + J_{n+2} \tag{7}$$

$$\frac{1}{\rho} J_n = \frac{1}{2n} \{J_{n-1} + J_{n+1}\}, \qquad J_{-1} = -J_1$$

and set the coefficients of J_n equal to zero, we obtain the following recursion relations for the coefficients, a_n :

$$a_1 = 2; \qquad a_2 = 4\eta a_1 \tag{8}$$

$$a_{n+1} = 4\eta(a_n/n) - a_{n-1} \qquad (n = 2, 3, 4, \cdots)$$

The convergence of (5) follows from the known absolute convergence of (2). To compare the convergence of the expansion (5) with the power series (2) we cite the following case. For $\rho = 5$, $\eta = 1$, we obtain $\rho\phi_0(\rho, \eta) = 6.317299$ with fifteen terms of (5) whereas thirty terms may be necessary using (2). It may be mentioned that solutions of (1) in terms of a series of Bessel Functions of half integral order have been given by Prof. P. M. Morse of M.I.T. in correspondence with the Computation Laboratory.

A technique similar to the above can be employed to obtain an expansion in terms of the modified Bessel Functions, $I_n(\rho)$. If we assume an expansion in the form

$$y = \sum_{n=1}^{\infty} b_n(\eta) I_n(\rho) \tag{9}$$

and make use of the relations

$$4\, I_n'' = I_{n-2} + 2\, I_n + I_{n+2}, \tag{10}$$

$$\frac{1}{\rho} I_n = \frac{1}{2n} (I_{n-1} - I_{n+1}), \qquad I_{-1} = I_1$$

we obtain the following recurrence relations:

$$b_1 = 2, \qquad b_2 = 2\eta b_1, \qquad b_3 = 2\eta b_2 - 7b_1,$$

$$b_4 = (4\eta/3)b_3 - 12\, b_2 - 4\eta b_1, \tag{11}$$

$$b_{n+1} = (4\eta/n)b_n - 6b_{n-1} - \{4\eta/(n-2)\}b_{n-2} - b_{n-3} \qquad (n = 3, 4, \cdots)$$

The convergence of the expansion (9) follows from the known absolute convergence of (2). However actual calculation has shown that the rate of convergence is similar to that of (5). The expansion in terms of $J_n(\rho)$ is superior to the one in terms of $I_n(\rho)$ because of the simpler form of the recurrence relations given in (8) as compared with those in (11).

THE COMPUTATION LABORATORY
NATIONAL BUREAU OF STANDARDS

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NOTE ON THE METHOD OF COMPLEMENTARY ENERGY

By Eric Reissner

In the following lines we propose to show that the well known minimum principle for the *stresses* in the theory of elasticity (the principle of minimum complementary energy) may be extended in such a way that it applies to dynamic as well as to static problems. While the proof of this extended principle is quite simple, we have not been able to find a statement of it in the literature. The subject is considered in quite a different way in a recent paper by H. M. Westergaard.[1]

We do not here intend to give the most general theorem possible[2] but rather restrict attention to problems in which loads, stresses and displacements are simple harmonic functions of time. We use the customary engineering notation for stresses (σ, τ) and displacements (u, v, w), and omit the common time factor $\cos \omega t$ in all equations of the theory. (In so doing we exclude consideration of materials with non-linear stress strain relations).

We have as differential equation of equilibrium in the x-direction

$$\partial\sigma_x/\partial x + \partial\tau_{xy}/\partial y + \partial\tau_{xz}/\partial z + X + \rho\omega^2 u = 0, \tag{1}$$

and two corresponding equations for equilibrium in the y and z-directions. As in the static case, which corresponds to $\omega = 0$, we have a stress energy function W which is determined by the relations

$$\partial W/\partial\sigma_x = \epsilon_x\,, \quad \partial W/\partial\tau_{xy} = \gamma_{xy}\,, \text{ etc.} \tag{2}$$

where the components of strain ϵ_x, γ_{xy} etc. are appropriately expressed in terms of the stresses σ_x, τ_{xy} etc., but are not at this stage required to satisfy the equations of compatibility.

We further introduce a kinetic energy function K given by

$$K = \tfrac{1}{2}\rho\omega^2(u^2 + v^2 + w^2) \tag{3}$$

If we assume for simplicity sake that on the entire surface of the body we have prescribed stresses then the extended theorem may be formulated as follows.

Theorem. Among all states of stress *and displacement* which satisfy the differential equations of equilibrium in the interior of the body and the condition of prescribed stress on the surface, the state of stress which also satisfies the compatibility relations for strain is determined by the variational equation

$$\delta \int (W - K)\, dV = 0.$$

[1] On the Method of Complementary Energy. Proc. American Soc. Civil Engs., February 1941, pp. 199–227.

[2] Such generalizations should also include the three-dimensional theory of elastic stability.

Proof. We have

$$\delta \int (W - K)\, dV = \int \left[\frac{\partial W}{\partial \sigma_x}\, \delta\sigma_x + \frac{\partial W}{\partial \tau_{xy}}\, \delta\tau_{xy} + \cdots - \rho\omega^2 (u\delta u + v\delta v + w\delta w) \right] dV \tag{4}$$

According to (2) the compatibility relations are equivalent to

$$\frac{\partial W}{\partial \sigma_x} = \frac{\partial u}{\partial x}, \qquad \frac{\partial W}{\partial \tau_{xy}} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \text{etc.} \tag{5}$$

The conditions that stresses and displacements satisfy the differential equations of equilibrium are of the form

$$\frac{\partial \delta\sigma_x}{\partial x} + \frac{\partial \delta\tau_{xy}}{\partial y} + \frac{\partial \delta\tau_{xz}}{\partial z} + \rho\omega^2\, \delta u = 0 \tag{6}$$

with corresponding equations involving δv and δw.

We put (5) into (4), integrate by parts, and take into account that because of the conditions of prescribed surface stress we have three relations of the form

$$\delta p_x = \delta\sigma_x \cos(n, x) + \delta\tau_{xy} \cos(n, y) + \delta\tau_{xz} \cos(n, z) = 0, \tag{7}$$

so that the remaining surface integrals vanish. Eq. (4) then assumes the form

$$\delta \int (W - K)\, dV = \int \left\{ u \left[\frac{\partial \delta\sigma_x}{\partial x} + \frac{\partial \delta\tau_{xy}}{\partial y} + \frac{\partial \delta\tau_{xz}}{\partial z} + \rho\omega^2\, \delta u \right] + v[\cdots] + w[\cdots] \right\} dV \tag{8}$$

The right hand side of (8) vanishes because of (6) and thus the theorem is proved.

Remark. As for the problem of statics a more general form of the theorem states that if over a part S_1 of the surface the stresses are prescribed while over the remaining part S_2 the displacement components have prescribed values $\bar{u}$, $\bar{v}$, $\bar{w}$ then the appropriate variational equation is

$$\delta \left\{ \int (W - K)\, dV - \int_{S_2} (\bar{u}p_x + \bar{v}p_y + \bar{w}p_z)\, dS \right\} = 0$$

where p_x, p_y and p_z are the unknown x, y, and z-components of the surface stress.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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THE DETERMINATION OF UPPER AND LOWER BOUNDS FOR THE SOLUTION OF THE DIRICHLET PROBLEM

By H. J. Greenberg

Introduction.[1] Stated in simple terms, the Dirichlet problem requires the determination of a harmonic function in a domain $\mathfrak{D}$, given only its values on the boundary of $\mathfrak{D}$. In this paper we present a method for determining upper and lower bounds for the value of this unknown function at any point of $\mathfrak{D}$. These bounds are expressed in terms of integrals involving arbitrary functions of two classes, U and V. The class U consists of all functions (defined in $\mathfrak{D}$ and possessing certain continuity properties) which satisfy the boundary conditions; the class V consists of all functions harmonic in $\mathfrak{D}$. The sharpness of the bounds obtained in any particular case, depends on which functions of U and V are selected; however, every choice of functions furnishes an upper bound and a lower bound. In the course of this paper an iterative procedure is developed, utilizing sequences of functions in U and V to obtain successively improved bounds for the value of the solution at any desired point.

We focus our attention upon one point of $\mathfrak{D}$ at a time, obtaining upper and lower bounds for the value of the solution at this point. This method is unique in that, having both an upper and lower bound for the exact solution at each point, *the maximum error at each point incurred by using any approximate intermediate value is known.* No such precise statement, concerning the maximum error at each point, can be made when one is using an approximate solution, obtained by standard methods such as the Rayleigh-Ritz method or Southwell's relaxation procedure.

The pointwise aspect of this method may be contrasted with the formal solution of the Dirichlet problem, given in terms of the Green's function for $\mathfrak{D}$. The present method, can be applied to domains for which the Green's function is not known. It appears, therefore, that by attempting to find the solution at one point at a time, rather than seeking at once the solution over the entire domain, the complexity of the problem is reduced to such an extent that actual numerical results can be obtained. In many problems in mathematical physics one is most interested in the value of the solution at particular points (points of maximum deflection, maximum stress, etc., where such points often can be determined à priori by symmetry or other considerations). Hence, the problem to be solved is not the determination of the solution over the whole domain, but the seemingly easier problem of finding the value of the solution at the point or points under consideration. It is for such problems that the method has been devised.

The method is based on the application of two complementary variational principles for the solution of the Dirichlet problem. One of these principles,

[1] Presented to the American Mathematical Society, February 22, 1947. A thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy at Brown University, October 1946.

the Dirichlet principle, applies to functions of the class U; the complementary principle applies to functions of the class V. The fact that the solution of the Dirichlet problem *maximizes* a certain expression, according to the first principle, while *minimizing*, a second expression, according to the complementary principle, leads to a set of fundamental inequalities on an integral of the solution. By introducing two auxiliary boundary-value problems we are able to deduce inequalities directly upon the value of the solution itself.

The results obtained hold for all sufficiently regular domains, the boundary values being continuous functions. The practical limitations on the method are (1) that one must, for the given domain and boundary values, be able to determine functions of the class U, and (2) that one must be able to integrate certain expressions. However, if (1) can be achieved, the integrations, can, at the very worst, always be carried through numerically, since the integrands involve only known functions.

The greater part of this paper is devoted to the application of the method to the Dirichlet problem in the plane. In Section 5 it is shown how the results can be immediately extended to three-space, and how the method may be applied to obtain upper and lower bounds directly for the derivatives of the solution of the Dirichlet problem at any point of the domain. Section 6, considers the homogeneous boundary-value problem associated with Poisson's equation, obtaining upper and lower bounds for the solution of this problem. The paper concludes with an example illustrating the method.

1. The complementary variational principles. Let $\mathcal{C}$ be a simple closed contour in the xy-plane, consisting of a finite number of regular arcs, and let $\mathcal{D}$ denote the domain interior to $\mathcal{C}$. Let s measure arc length along $\mathcal{C}$ and n be the inwardly directed unit normal to $\mathcal{C}$. Assume that we are given a continuous function $f(s)$ defined on $\mathcal{C}$ and wish to solve the Dirichlet problem of determining a function $w(x, y)$ such that

$$\Delta w \equiv \partial^2 w/\partial x^2 + \partial^2 w/\partial y^2 = 0, \text{ in } \mathcal{D}, \tag{1.1}$$

$$w = f, \text{ on } \mathcal{C}. \tag{1.2}$$

We shall designate the Dirichlet problem (1.1), (1.2) as problem I. Under the conditions which we have imposed on $\mathcal{D}$, $\mathcal{C}$ and f it is known that there exists a unique solution w to this problem. It is our purpose to find upper and lower bounds for the value of w at an arbitrary point of $\mathcal{D}$. We shall proceed to formulate two variational principles for the function w.

The Dirichlet principle. Let U be the class of all functions u which assume on $\mathcal{C}$ the given values f.[2] Then, for any $u \in U$,

$$D(u) \leq D(w), \tag{1.3}$$

[2] The functions of U as well as of all other classes to be introduced in this paper are assumed to be defined and required to be continuous throughout $\mathcal{D} + \mathcal{C}$ and to possess continuous derivatives of the second order in $\mathcal{D}$.

where w is the function defined by (1.1) and (1.2), and

$$D(u) = -\frac{1}{2}\int_{\mathfrak{D}} (u_x^2 + u_y^2)\, dA.^3 \tag{1.4}$$

In addition, $D(w) = D(u)$ if and only if $u = w$. It will be noticed that since we define $D(u)$ with the minus sign, $D(w)$ is an absolute maximum instead of the absolute minimum of the usual formulation.

The complementary principle. Let V be the class of all functions v which satisfy in $\mathfrak{D}$ Laplace's equation

$$\Delta v = 0. \tag{1.5}$$

Then for any $v \in V$

$$D^*(w) \leq D^*(v), \tag{1.6}$$

where w is the function defined by (1.1) and (1.2), and

$$D^*(v) = \frac{1}{2}\int_{\mathfrak{D}} (v_x^2 + v_y^2)\, dA + \int_{\mathfrak{C}} f \frac{\partial v}{\partial n}\, ds. \tag{1.7}$$

In addition, $D^*(v) = D^*(w)$ if and only if $v = w + c$, where c is a constant.

This principle, was obtained by Friedrichs[4] from the Dirichlet principle by applying a Legendre type transformation. A direct proof of the principle is easily given. Defining $v - w = w'$ we have

$$\begin{aligned} D^*(v) = D^*(w) &+ \frac{1}{2}\int_{\mathfrak{D}} (w_x'^2 + w_y'^2)\, dA \\ &+ \int_{\mathfrak{C}} f \frac{\partial w'}{\partial n}\, ds + \int_{\mathfrak{D}} (w_x' w_x + w_y' w_y)\, dA. \end{aligned} \tag{1.8}$$

Since w' is a harmonic function, the sum of the last two terms in (1.8) is zero by Green's first identity[5]

$$\int_{\mathfrak{C}} \phi \frac{\partial x}{\partial n}\, ds + \int_{\mathfrak{D}} (\phi_x \psi_x + \phi_y \psi_y)\, dA = -\int_{\mathfrak{D}} \phi \Delta \psi\, dA.$$

It follows that $D^*(v) \geq D^*(w)$ and that equality holds if and only if $v = w + c$, c a constant. The fact that $D^*(v) = D^*(w)$ does not imply $v = w$ seems not to have been emphasized. This fact is in contrast to Dirichlet's principle where $D(u) = D(w)$ implies $u = w$. However, if in place of V we take the subclass V'

[3] We use the subscripts x and y to indicate partial differentiations with respect to these variables.

[4] Friedrichs, K. O., *Ein Verfahren der Variationsrechnung, das Minimum eines Integrals als das Maximum eines Anderen Ausdruckes darzustellen*, Nachrichten der Ges. d. Wiss. zu Göttingen, 1929, pp. 13–20.

[5] The terminology is that of O. D. Kellogg, *Foundations of potential theory*, New York, 1929, p. 212.

of functions v' each of which is contained in V and which in addition satisfy the requirement that $v' = f$ at at least one point of $\mathcal{C}$, then $D^*(v') = D^*(w)$ implies $v' = w$. Hence, the solution w of I renders D^* an *absolute* minimum over the class V'. We note in passing that taking a function v' in V' plus a linear combination of functions in V which vanish at the point of $\mathcal{C}$ at which v' agrees with f, and applying the operator D^*, w can be approximated by a procedure analogous to the Rayleigh-Ritz method.

A final remark serves to emphasize the complementary character of the two variational principles we have been discussing. The Dirichlet principle considers the class of functions U, satisfying the boundary conditions (1.2), the other principle considers the class of functions V (or V') satisfying the partial differential equation (1.1). There is only one function that is common to both U and V and that is $w(x, y)$, the solution of the Dirichlet problem (1.1), (1.2). Thus, U and V may be looked upon as defining a sort of Dedekind cut in function space, this cut corresponding to the function $w(x, y)$.

2. The fundamental inequalities. The following relation between the maximum value of the Dirichlet integral D and the minimum value of the complementary "integral" D^* follows at once from Green's first identity:

$$D(w) = D^*(w) = \frac{1}{2}\int_{\mathcal{C}} f \frac{\partial w}{\partial n}\, ds, \tag{2.1}$$

where w is the function defined by (1.1), (1.2). A comparison of (1.3), (1.6) and (2.1) yields the following theorem:

Theorem 1. The fundamental inequalities. For any $u \in U$ and $v \in V$

$$-\frac{1}{2}\int_{\mathfrak{D}} (u_x^2 + u_y^2)\, dA \leq \frac{1}{2}\int_{\mathcal{C}} f \frac{\partial w}{\partial n}\, ds \leq \int_{\mathcal{C}} f \frac{\partial v}{\partial n}\, ds + \frac{1}{2}\int_{\mathfrak{D}} (v_x^2 + v_y^2)\, dA, \tag{2.2}$$

where w is the function defined by (1.1), (1.2).

The content of this theorem is not new. It expresses the fact that using the principles of Dirichlet and Friedrichs, $D(w)$ can be bounded from above and below. As such it has been used by previous investigators to give bounds for the over-all error committed by using an approximating function $u \in U$ obtained by Ritz' method, i.e. $| D(u) - D(w) | \leq | D(u) - D^*(v) |$, where v is any harmonic function. See for instance N. M. Basu, *On an application of the new methods of the calculus of variations to some problems in the theory of elasticity*, Phil. Mag., ser. 7, 10, 1930, pp. 886–896. The formulation of (2.2) seems to be more explicit, however, and leads us eventually to upper and lower bounds directly on the value of the solution w at any individual point of the domain, in contrast to upper and lower bounds on $D(w)$ which give no information as to the value of the solution at individual points.

It is convenient for future reference to introduce the nototations $L_1(u)$, W_1, $U_1(v)$ for the lower bound, middle term and upper bound respectively in (2.2) so that (2.2) becomes simply

$$L_1(u) \leq W_1 \leq U_1(v). \tag{2.3}$$

We note that L_1 is in any case negative and we observe from (2.1) that the value of W_1 is also negative. We inquire as to the sign of U_1. Note that if $v \in V$, $\lambda v \in V$ where λ is an arbitrary real constant. Hence $U_1(\lambda v)$ is an upper bound by (2.2). Provided that v is not a constant we may minimize this expression with respect to λ to obtain as a corollary to Theorem 1

$$(2.4) \qquad -\frac{1}{2}\int_{\mathfrak{D}} (u_x^2 + u_y^2)\, dA \leq \frac{1}{2}\int_{\mathfrak{C}} f\frac{\partial w}{\partial n}\, ds \leq -\frac{\left(\int_{\mathfrak{C}} f\frac{\partial v}{\partial n}\, ds\right)^2}{2\int_{\mathfrak{D}} (v_x^2 + v_y^2)\, dA},$$

showing that the upper bound in (2.2) for a non-constant $v \in V$ can always be made negative and hence non-trivial. It will be seen that this device for improving the upper bound is simply the first step of an iterative process which can be applied to both the upper and lower bounds to progressively improve the estimates. The question of iteration is discussed at length in Section 4.

3. Upper and lower bounds for the solution. We propose now to find upper and lower bounds on $w(x, y)$ at a *particular interior* point (x_0, y_0) of $\mathfrak{D}$. Let $r = [(x - x_0)^2 + (y - y_0)^2]^{\frac{1}{2}}$ be the distance from (x_0, y_0) to a variable point (x, y) of $\mathfrak{D} + \mathfrak{C}$. We consider the following boundary-value problem.

$$(3.1) \qquad \Delta\bar{w} = 0, \text{ in } \mathfrak{D},$$

$$(3.2) \qquad \bar{w} = -P \log 1/r, \text{ on } \mathfrak{C},$$

for $\bar{w}(x, y)$ and refer to it henceforth as problem II; P is a real constant which will later play the role of a parameter. We note that since (x_0, y_0) is an interior point of $\mathfrak{D}$, the function $-P \log 1/r$ is continuous on the boundary of $\mathfrak{D}$. Hence $\bar{w}(x, y)$ exists and is a regular function.

We now apply Theorem 1 to II, which is a problem of the same type as I. Replacing f by $-P \log 1/r$ in (2.2) we get

$$(3.3) \qquad \begin{aligned} -\frac{1}{2}\int_{\mathfrak{D}} (\bar{u}_x^2 + \bar{u}_y^2)\, dA &\leq -\frac{P}{2}\int_{\mathfrak{C}} \left(\log\frac{1}{r}\right)\frac{\partial\bar{w}}{\partial n}\, ds \\ &\leq -P\int_{\mathfrak{C}} \left(\log\frac{1}{r}\right)\frac{\partial\bar{v}}{\partial n}\, ds + \frac{1}{2}\int_{\mathfrak{D}} (\bar{v}_x^2 + \bar{v}_y^2)\, dA. \end{aligned}$$

where $\bar{v} \in V$ and $\bar{u}(x, y)$ may be any function such that $\bar{u} = -P \log 1/r$ on $\mathfrak{C}$. Thus, both $\bar{v}$ in (3.3) and the function v in (2.3) are subject to exactly the same restriction and hence might be chosen to be the same function. In general, however we want to be able to choose v and $\bar{v}$ to be different functions; this accounts for the use of separate notations. More explicitly, we are interested in making our inequalities as sharp as possible. We know that if $\bar{v}$ happens to be $\bar{w}$ that the right side of (3.3) reduces to an equality. Thus, when choosing functions $\bar{v}$, we seek to approximate $\bar{w}$ and similarly when choosing functions v, we seek to approximate w. Since $\bar{w}$ and w are the solutions of two different boundary value problems we must allow for $\bar{v}$ and v to be quite different functions although both must be harmonic.

Recalling now a familiar theorem of potential theory, we have the following equation giving the harmonic function w in terms of its boundary values and those of its normal derivatives

$$2\pi\bar{w}(x_0, y_0) = -\int_c \left(\log\frac{1}{r}\right)\frac{\partial\bar{w}}{\partial n}\,ds - P\int_c \left(\log\frac{1}{r}\right)\frac{\partial(\log 1/r)}{\partial n}\,ds. \tag{3.4}$$

This equation enables us to rewrite the central member of the continued inequality (3.3). We obtain

$$\begin{aligned} -\frac{1}{2}\int_{\mathfrak{D}} (\bar{u}_x^2 + \bar{u}_y^2)\,dA &\leq \pi P\bar{w}(x_0, y_0) + \frac{P^2}{2}\int_c \left(\log\frac{1}{r}\right)\frac{\partial(\log 1/r)}{\partial n}\,ds \\ &\leq -P\int_c \left(\log\frac{1}{r}\right)\frac{\partial\bar{v}}{\partial n}\,ds + \frac{1}{2}\int_{\mathfrak{D}} (\bar{v}_x^2 + \bar{v}_y^2)\,dA. \end{aligned} \tag{3.5}$$

This result gives us upper and lower bounds for $\bar{w}(x_0, y_0)$, since the quantity

$$\frac{P^2}{2}\int_c \left(\log\frac{1}{r}\right)\frac{\partial(\log 1/r)}{\partial n}\,ds \tag{3.6}$$

is a constant (the point (x_0, y_0) is to be considered as fixed throughout the remainder of this discussion) whose value can be computed without knowing the function $\bar{w}$. Subtracting this quantity from the three members of (3.5) we obtain the inequality

$$L_2 \leq W_2 \leq U_2 \tag{3.7}$$

where

$$L_2(\bar{u}) = -\frac{1}{2}\int_{\mathfrak{D}} (\bar{u}_x^2 + \bar{u}_y^2)\,dA - \frac{P^2}{2}\int_c \left(\log\frac{1}{r}\right)\frac{\partial(\log 1/r)}{\partial n}\,ds, \tag{3.8}$$

$$W_2 = \pi P\bar{w}(x_0, y_0), \tag{3.9}$$

$$\begin{aligned} U_2(\bar{v}) = -P\int_c \left(\log\frac{1}{r}\right)\frac{\partial\bar{v}}{\partial n}\,ds + \frac{1}{2}\int_{\mathfrak{D}} (\bar{v}_x^2 + \bar{v}_y^2)\,dA \\ - \frac{P^2}{2}\int_c \left(\log\frac{1}{r}\right)\frac{\partial(\log 1/r)}{\partial n}\,ds. \end{aligned} \tag{3.10}$$

So far we have obtained two sets of inequalities, namely (2.3) and (3.7). We shall require a third set of inequalities in addition to these two. Before proceeding with the derivation of this third set, however, it is worthwhile to consider (3.7) by itself. We recall the definition of the Green's function for the domain $\mathfrak{D}$:

$$G(x, y; x_0, y_0) = \log 1/r + g(x, y; x_0, y_0) \tag{3.11}$$

where

$$\Delta g = 0, \text{ in } \mathfrak{D}, \qquad g = -\log 1/r, \text{ on } \mathcal{C}.$$

Since the function g is just the function $\overline{w}(x, y)$ of II when $P = 1$, the inequalities (3.7) can be written

$$-\frac{1}{2\pi}\left[\int_{\mathfrak{D}} (\bar{u}_x^2 + \bar{u}_y^2)\, dA + \int_{\mathfrak{C}} \left(\log\frac{1}{r}\right)\frac{\partial(\log 1/r)}{\partial n}\, ds\right]$$

$$\leq g(x_0, y_0; x_0, y_0) \leq \tag{3.12}$$

$$\frac{1}{2\pi}\left[-2\int_{\mathfrak{C}} \left(\log\frac{1}{r}\right)\frac{\partial \bar{v}}{\partial n}\, ds + \int_{\mathfrak{D}} (\bar{v}_x^2 + \bar{v}_y^2)\, dA - \int_{\mathfrak{C}} \left(\log\frac{1}{r}\right)\frac{\partial(\log 1/r)}{\partial n}\, ds\right].$$

This result gives upper and lower bounds at (x_0, y_0) for the regular function g associated with the Green's function G. Numerical values for these bounds can be computed for each choice of an admissible function $\bar{u}$, i.e. assuming the boundary values $-\log 1/r$ on $\mathfrak{C}$, and an admissible function $\bar{v}$, i.e. $\bar{v} \in V$. Since g is merely a special harmonic function, later results giving bounds on w may be applied to yield bounds for g at points other than (x_0, y_0), the pole of the Green's function.

Returning now to the main discussion we proceed to set up the fundamental inequalities which correspond to the boundary value problem

$$\Delta\omega = 0, \text{ in, } \mathfrak{D} \tag{3.13}$$

$$\omega = f - P \log 1/r, \text{ on } \mathfrak{C}, \tag{3.14}$$

which we shall designate as problem III. The procedure is clear. We apply Theorem 1 to III which is again a problem of the same type as I. In (2.3) this means we must replace f by $f - P \log 1/r$ and u by a function μ which assumes on $\mathfrak{C}$ the values $f - P \log 1/r$; in place of v we shall write ν where ν, like v, may be any function contained in V, but which, in general, we allow to be distinct from v. Carrying through these substitutions we get

$$-\frac{1}{2}\int_{\mathfrak{D}} (\mu_x^2 + \mu_y^2)\, dA$$

$$\leq \frac{1}{2}\int_{\mathfrak{C}} f\frac{\partial\omega}{\partial n}\, ds - \frac{P}{2}\int_{\mathfrak{C}} \left(\log\frac{1}{r}\right)\frac{\partial\omega}{\partial n}\, ds \leq \tag{3.15}$$

$$\int_{\mathfrak{C}} f\frac{\partial\nu}{\partial n}\, ds - P\int_{\mathfrak{C}} \left(\log\frac{1}{r}\right)\frac{\partial\nu}{\partial n}\, ds + \frac{1}{2}\int_{\mathfrak{D}} (\nu_x^2 + \nu_y^2)\, dA.$$

As in the derivation of (3.7) we now use the representation theorem for the harmonic function ω in terms of its boundary values and those of its normal derivatives, viz.,

$$2\pi\omega(x_0, y_0) = \int_{\mathfrak{C}} \left(f - P\log\frac{1}{r}\right)\frac{\partial(\log 1/r)}{\partial n}\, ds - \int_{\mathfrak{C}} \left(\log\frac{1}{r}\right)\frac{\partial\omega}{\partial n}\, ds. \tag{3.16}$$

Solving for the last integral in (3.16) and substituting in (3.15) we find

$$-\frac{1}{2}\int_{\mathfrak{D}} (\mu_x^2 + \mu_y^2)\, dA \leq \frac{1}{2}\int_{\mathfrak{C}} f\frac{\partial\omega}{\partial n}\, ds + \pi P\omega(x_0, y_0)$$

$$-\frac{P}{2}\int_e f\frac{\partial(\log 1/r)}{\partial n}\,ds+\frac{P^2}{2}\int_e\left(\log\frac{1}{r}\right)\frac{\partial(\log 1/r)}{\partial n}\,ds \tag{3.17}$$

$$\leq\int_e f\frac{\partial\nu}{\partial n}\,ds-P\int_e\left(\log\frac{1}{r}\right)\frac{\partial\nu}{\partial n}\,ds+\frac{1}{2}\int_{\mathfrak{D}}(\nu_x^2+\nu_y^2)\,dA.$$

In view of the linearity of problems I, II and III,

$$\omega=w+\overline{w}. \tag{3.18}$$

In choosing admissible functions μ and ν we shall take advantage of this fact by setting

$$\mu=u+\bar{u},\qquad \nu=v+\bar{v}. \tag{3.19}$$

Using (3.18) we rewrite the terms involving ω in (3.17)

$$P\pi\omega(x_0,y_0)=P\pi w(x_0,y_0)+P\pi\overline{w}(x_0,y_0), \tag{3.20}$$

$$\frac{1}{2}\int_e f\frac{\partial\omega}{\partial n}\,ds=\frac{1}{2}\int_e f\frac{\partial w}{\partial n}\,ds+\frac{1}{2}\int_e f\frac{\partial\overline{w}}{\partial n}\,ds. \tag{3.21}$$

It will be observed in (3.20) and (3.21) that $w(x_0,y_0)$ is the quantity for which we are attempting to find bounds; upper and lower bounds for $P\pi w(x_0,y_0)$ and $\frac{1}{2}\int_e f(\partial w/\partial n)\,ds$ were obtained in (3.7) and (2.3) respectively. The term $\frac{1}{2}\int_e f(\partial\overline{w}/\partial n)\,ds$ in (3.21) which involves the boundary value f of problem I and the normal derivatives of the solution $\overline{w}$ of problem II is disposed of as follows. Green's formula for the solution of I in terms of the Green's function of (3.11) is simply

$$w(x_0,y_0)=\frac{1}{2\pi}\int_e f\frac{\partial}{\partial n}\left[\log\frac{1}{r}+g\right]ds. \tag{3.22}$$

Noting that $Pg\equiv\overline{w}$ we deduce that

$$\frac{1}{2}\int_e f\frac{\partial\overline{w}}{\partial n}\,ds=\pi Pw(x_0,y_0)-\frac{P}{2}\int_e f\frac{\partial(\log 1/r)}{\partial n}\,ds. \tag{3.23}$$

Substituting (3.23) into (3.21) and (3.21) and (3.20) into (3.17) and shifting terms we arrive at the third set of inequalities

$$L_3\leq W_3\leq U_3 \tag{3.24}$$

where

$$L_3=-\frac{1}{2}\int_{\mathfrak{D}}(\mu_x^2+\mu_y^2)\,dA$$

$$+P\int_e f\frac{\partial(\log 1/r)}{\partial n}\,ds-\frac{P^2}{2}\int_e\left(\log\frac{1}{r}\right)\frac{\partial(\log 1/r)}{\partial n}\,ds, \tag{3.25}$$

$$W_3=2\pi Pw(x_0,y_0)+\frac{1}{2}\int_e f\frac{\partial w}{\partial n}\,ds+\pi P\overline{w}(x_0,y_0), \tag{3.26}$$

$$U_3 = \int_{\mathcal{C}} f \frac{\partial v}{\partial n}\, ds - P \int_{\mathcal{C}} \left(\log \frac{1}{r}\right) \frac{\partial v}{\partial n}\, ds + \frac{1}{2} \int_{\mathfrak{D}} (v_x^2 + v_y^2)\, dA$$

(3.27)

$$+ P \int_{\mathcal{C}} f \frac{\partial (\log 1/r)}{\partial n}\, ds - \frac{P^2}{2} \int_{\mathcal{C}} \left(\log \frac{1}{r}\right) \frac{\partial (\log 1/r)}{\partial n}\, ds.$$

Comparing W_3 with the quantities W_1 and W_2 previously defined we see that

$$W_3 = 2\pi P w(x_0, y_0) + W_1 + W_2. \tag{3.28}$$

Now, (2.3), (3.7) and (3.24) provide bounds for W_1, W_2, and W_3 respectively. It follows, by (3.28), that bounds are thereby determined for the quantity $2\pi P w(x_0, y_0)$. We have at once the result summarized in the following theorem.

Theorem 2.

$$\mathfrak{L} \leq 2\pi P w(x_0, y_0) \leq \mathfrak{U}, \tag{3.29}$$

where

$$\mathfrak{L} \equiv L_3 - U_1 - U_2 = -\frac{1}{2} \int_{\mathfrak{D}} (\mu_x^2 + \mu_y^2)\, dA + P \int_{\mathcal{C}} f \frac{\partial (\log 1/r)}{\partial n}\, ds$$

$$- \int_{\mathcal{C}} f \frac{\partial v}{\partial n}\, ds - \frac{1}{2} \int_{\mathfrak{D}} (v_x^2 + v_y^2)\, dA \tag{3.30}$$

$$+ P \int_{\mathcal{C}} \left(\log \frac{1}{r}\right) \frac{\partial \bar{v}}{\partial n}\, ds - \frac{1}{2} \int_{\mathfrak{D}} (\bar{v}_x^2 + \bar{v}_y^2)\, dA,$$

$$U_3 - L_1 - L_2 = \int_{\mathcal{C}} f \frac{\partial \nu}{\partial n}\, ds - P \int_{\mathcal{C}} \left(\log \frac{1}{r}\right) \frac{\partial \nu}{\partial n}\, ds$$

$$+ \frac{1}{2} \int_{\mathfrak{D}} (\nu_x^2 + \nu_y^2)\, dA + P \int_{\mathcal{C}} f \frac{\partial (\log 1/r)}{\partial n}\, ds \tag{3.31}$$

$$+ \frac{1}{2} \int_{\mathfrak{D}} (u_x^2 + u_y^2)\, dA + \frac{1}{2} \int_{\mathfrak{D}} (\bar{u}_x^2 + \bar{u}_y^2)\, dA.$$

in which $u, v, \bar{u}, \bar{v}, \mu, \nu$ are functions satisfying the conditions:

1. $u \in U$, 2. $\bar{u} = -P \log 1/r$ on $\mathcal{C}$,

3. $v \in V$, $\bar{v} \in V$, 4. $\mu = u + \bar{u}$, $\nu = v + \bar{v}$,

and P is a constant, arbitrary but fixed.

The above theorem establishes the desired upper and lower bounds for the solution w of I at the point (x_0, y_0) of $\mathfrak{D}$. In order to apply the theorem in a particular problem requires the selection of several functions. To find a function u may or may not be difficult depending on both $\mathcal{C}$ and f. There is, of course, no difficulty at all in finding functions in V. However, a word should be said about the choice of a function $\bar{u}$. We note that $-P \log 1/r$ can not be used due to the fact that the singularity of this function at (x_0, y_0) causes $\int_{\mathfrak{D}} (\bar{u}_x^2 + \bar{u}_y^2)\, dA$ to diverge. However, if we properly truncate the function

$-P \log 1/r$ in the neighborhood of (x_0, y_0) so as to eliminate the singularity, we obtain a function which can be taken for $\bar{u}$. Let C_R be a circle of radius R about (x_0, y_0), lying entirely in $\mathfrak{D}$. Then we may choose $\bar{u}$ to be the function

$$\bar{u} = \begin{cases} -P \log 1/r, & \text{at points of } \mathfrak{D} \text{ outside } C_R, \\ ar^2 + br + c, & \text{inside } C_R, \end{cases} \tag{3.32}$$

where we require $\bar{u}$ to have continuous second derivatives with respect to r across the circle C_R. This determines the coefficients a, b, c to be

$$a = -\frac{P}{2R^2}, \qquad b = \frac{2P}{R}, \qquad c = -P\left(\log \frac{1}{R} + \frac{3}{2}\right). \tag{3.33}$$

To obtain additional admissible functions we may add to the truncated function (3.32) any function contained in $C^{(2)}$ in $\mathfrak{D}$ which vanishes on $\mathfrak{C}$.

We now consider the real parameter P which has appeared throughout the discussion and which is present in the final inequalities (3.29). We shall see that the presence of this parameter enables us to get an improved set of bounds for $w(x_0, y_0)$. We note that the conditions of Theorem 2 are complied with if

$$\bar{u} = P\dot{u}, \tag{3.34}$$

$$\bar{v} = P\dot{v} \tag{3.35}$$

where $\dot{u} = -\log 1/r$ on $\mathfrak{C}$ and $\dot{v} \in V$. We note too that the truncated function (3.32), for example, is of the form (3.34).

Making the substitutions (3.34) and (3.35) in (3.30) and (3.31), and setting $\mu = u + P\dot{u}$, $\nu = v + P\dot{v}$ we find that both $\mathfrak{L}$ and $\mathfrak{A}$ are quadratics in P. Thus,

$$\mathfrak{L} = AP^2 + BP + C, \tag{3.36}$$

$$\mathfrak{A} = -AP^2 + B'P - C, \tag{3.37}$$

where

$$\begin{aligned} A &= -\frac{1}{2}\int_{\mathfrak{D}} (\dot{u}_x^2 + \dot{u}_y^2)\, dA + \int_{\mathfrak{C}} \left(\log \frac{1}{r}\right) \frac{\partial \dot{v}}{\partial n}\, ds - \frac{1}{2}\int_{\mathfrak{D}} (\dot{v}_x^2 + \dot{v}_y^2)\, dA, \\ B &= -\int_{\mathfrak{D}} (u_x \dot{u}_x + u_y \dot{u}_y)\, dA + \int_{\mathfrak{C}} f\, \frac{\partial\, (\log 1/r)}{\partial n}\, ds, \\ C &= -\frac{1}{2}\int_{\mathfrak{D}} (u_x^2 + u_y^2)\, dA - \int_{\mathfrak{C}} f\, \frac{\partial v}{\partial n}\, ds - \frac{1}{2}\int_{\mathfrak{D}} (v_x^2 + v_y^2)\, dA, \\ B' &= \int_{\mathfrak{D}} (v_x \dot{v}_x + v_y \dot{v}_y)\, dA + \int_{\mathfrak{C}} f\, \frac{\partial\, (\log 1/r)}{\partial n}\, ds \\ &\qquad + \int_{\mathfrak{C}} f\, \frac{\partial \dot{v}}{\partial n}\, ds - \int_{\mathfrak{C}} \left(\log \frac{1}{r}\right) \frac{\partial v}{\partial n}\, ds. \end{aligned} \tag{3.38}$$

If in addition to the above notations, we write $m = 2\pi w(x_0, y_0)$, (3.29) becomes

$$AP^2 + BP + C \leq mP \leq -AP^2 + B'P - C. \tag{3.39}$$

We now observe that (3.39) must hold for any value of the real parameter P,

since we are free to assign any value to P in problem II. Therefore

$$AP^2 + (B - m)P + C \leq 0, \tag{3.40}$$

and

$$-AP^2 + (B' - m)P - C \geq 0, \tag{3.41}$$

for any value of P and hence are definite quadratic forms. It follows that

$$B - 2\sqrt{AC} \leq m \leq B + 2\sqrt{AC}, \tag{3.42}$$

$$B' - 2\sqrt{AC} \leq m \leq B' + 2\sqrt{AC}, \tag{3.43}$$

These results are summarized in the following theorem.

Theorem 3. The value $2\pi w(x_0, y_0)$ must be contained in each of the intervals (3.42), (3.43), where A, B, C, B' are the quantities of (3.38) evaluated for any set of functions $u, \dot{u}, v, \dot{v}$ satisfying the following conditions:

1. $u \in U$. 2. $\dot{u} = -\log 1/r$ on $\mathcal{C}$. 3. $v \in V$, $\dot{v} \in V$.

Before concluding this section we make a final remark. Suppose in I the function f were $-\log [(x - x_0)^2 + (y - y_0)^2]^{\frac{1}{2}}$. The solution of I would then be, in the notation of (3.11), $g(x, y; x_0, y_0)$ the regular function associated with the Green's function $G(x, y; x_0, y_0)$. By virtue of Theorem 3 we can obtain upper and lower bounds on $g(\xi, \eta; x_0, y_0)$ where (ξ, η) is any point of $\mathfrak{D}$. It should be noted that in formulas (3.38) r becomes the distance from (ξ, η) and that f is replaced by $-\log [(x - x_0)^2 + (y - y_0)^2]^{\frac{1}{2}}$. Bounds for the special case where (ξ, η) is the pole (x_0, y_0) of the Green's function were previously given in (3.12).

4. An iteration procedure. By Theorem 3 it is known that the value of $m = 2\pi w(x_0, y_0)$ must lie in an interval K, the intersection of (3.42) and (3.43), whose length is determined by the values A, B, C, B' of (3.38). These quantities are completely determined by a choice set of four admissible functions $u, \dot{u}, v, \dot{v}$. The question naturally arises whether it is possible to construct a sequence of such sets which provides successively improved bounds for m. We shall show that this can be done. More precisely, we shall construct a sequence of sets $u_n, \dot{u}_n, v_n, \dot{v}_n$, $n = 0, 1, 2, \cdots$, having the property that $K_{n-1} \supseteq K_n$, $n = 1, 2, \cdots$, where K_n is the interval for m determined by the n'th set of the sequence.

To begin, we must have an initial set of functions $u = u_0$, $v = v_0$, $\dot{u} = \dot{u}_0$, $\dot{v} = \dot{v}_0$ satisfying the admissibility conditions

$$u_0 = f, \text{ on } \mathcal{C},$$

$$\Delta v_0 = \Delta \dot{v}_0 = 0, \text{ in } \mathfrak{D},$$

$$\dot{u}_0 = -\log 1/r, \text{ on } \mathcal{C}.$$

Denoting by A_0, B_0, C_0, B_0' quantities of (3.38) evaluated for these functions, we have by (3.39)

$$A_0P^2 + B_0P + C_0 \leq mP \leq -A_0P^2 + B_0'P - C_0. \tag{4.1}$$

By Theorem 3 it follows that

$$B_0 - 2\sqrt{A_0C_0} \leq m \leq B_0 + 2\sqrt{A_0C_0} \tag{4.2}$$

$$B_0' - 2\sqrt{A_0C_0} \leq m \leq B_0' + 2\sqrt{A_0C_0} \tag{4.3}$$

and hence that m lies in the intersection, or common part of the interval (4.2) and (4.3).

If, in addition to the initial functions we also have functions u_1' and v_1', satisfying the conditions

$$u_1' = 0, \text{ on } \mathcal{C}, \qquad \Delta v_1' = 0, \text{ in } \mathcal{D},$$

we can construct the new set of admissible functions

$$\begin{aligned} u_1 &= u_0 + g_1u_1', \qquad \dot{u}_1 = \dot{u}_0 + h_1u_1', \\ v_1 &= v_0 + k_1v_1', \qquad \dot{v}_1 = \dot{v}_0 + l_1v_1', \end{aligned} \tag{4.4}$$

where g_1, h_1, k_1, l_1, are constants to be determined. Applying (3.39) to these new functions we have

$$A_1P^2 + B_1P + C_1 \leq mP \leq -A_1P^2 + B_1'P - C_1 \tag{4.5}$$

where A_1, B_1, C_1, B_1' are the quantities of (3.38) evaluated for the functions defined by (4.4). By Theorem 3 we then have

$$B_1 - 2\sqrt{A_1C_1} \leq m \leq B_1 + 2\sqrt{A_1C_1} \tag{4.6}$$

$$B_1' - 2\sqrt{A_1C_1} \leq m \leq B_1' + 2\sqrt{A_1C_1} \tag{4.7}$$

and consequently m must lie in the common part of the intervals (4.6) and (4.7).

Substituting (4.4) into (3.8) we find that A_1 is the sum of a quadratic in h_1 and a quadratic in l_1, while C_1 is the sum of a quadratic in g_1 and a quadratic in k_1. By proper choice of these constants, each of these quadratics and consequently A_1 and C_1 can be maximized. Using the notations

$$\begin{aligned} H(X) &= \frac{1}{2}\int_{\mathcal{D}} (X_x^2 + X_y^2)\, dA, \\ H(X, Y) &= \frac{1}{2}\int_{\mathcal{D}} (X_xY_x + X_yY_y)\, dA \end{aligned} \tag{4.8}$$

the constants are determined to be

$$\begin{aligned} g_1 &= -\frac{H(u_0, u_1')}{H(u_1')}, \qquad h_1 = -\frac{H(\dot{u}_0, u_1')}{H(u_1')}, \\ l_1 &= -\frac{\left[H(\dot{v}_0, v_1') - \frac{1}{2}\int_e \left(\log\frac{1}{r}\right)\frac{\partial v_1'}{\partial n}\, ds\right]}{H(v_1')}, \\ k_1 &= -\frac{\left[\frac{1}{2}\int_e f\frac{\partial v_1'}{\partial n}\, ds + H(v_0, v_1')\right]}{H(v_1')} \end{aligned} \tag{4.9}$$

(assuming u_1' and v_1' to be chosen so that $H(u_1') \neq 0, H(v_1') \neq 0$). Using the values (4.9), A_1, B_1, C_1, B_1' become

$$A_1 = A_0 + \frac{[H(u_0, u_1')]^2}{H(u_1')} + \frac{\left[H(v_0, v_1') + \frac{1}{2}\int_e \left(\log\frac{1}{r}\right)\frac{\partial v_1'}{\partial n}\,ds\right]^2}{H(v_1')},$$

$$C_1 = C_0 + \frac{[H(u_0, u_1')]^2}{H(u_1')} + \frac{\left[\frac{1}{2}\int_e f\frac{\partial v_1'}{\partial n}\,ds - H(v_0, v_1')\right]^2}{H(v_1')},$$

(4.10)

$$B_1 = B_0 + 2\frac{H(u_0, u_1')\cdot H(u_0, u_1')}{H(u_1')},$$

$$B_1' = B_0' - 2\frac{\left[H(v_0, v_1') + \frac{1}{2}\int_e f\frac{\partial v_1'}{\partial n}\,ds\right]\left[H(v_0, v_1') - \frac{1}{2}\int_e \log\left(\frac{1}{r}\right)\frac{\partial v_1'}{\partial n}\,ds\right]}{H(v_1')}.$$

The formulas (4.10) determine the bounds (4.6), (4.7) for m. We shall show next that by virtue of the way in which the functions (4.4) have been chosen, the bounds (4.6), (4.7) for m are better than the initial estimate (4.2) (4.3). We remark that once the constants (4.9) have been evaluated, the values of all integrals which enter into formulas (4.10) are known so that no additional integrations are required.

From (4.10) we can show that

$$(B_1 - B_0)^2 \leq 4(A_1 - A_0)(C_1 - C_0).$$

Moreover, $C_0 \leq C_1$ and so

$$A_0P^2 + B_0P + C_0 \leq A_1P^2 + B_1P + C_1 \tag{4.11}$$

is valid for all values of P. In the same way we get

$$(B_1' - B_0')^2 \leq 4(A_1 - A_0)(C_1 - C_0)$$

and

$$-A_1P^2 + B_1'P - C_1 \leq -A_0P^2 + B_0P - C_0 \tag{4.12}$$

for all values of P. Combining (4.1), (4.5), (4.11), (4.12) we find that

$$\begin{aligned} A_0P^2 + B_0P + C_0 \leq A_1P^2 + B_1P + C_1 \leq mP \leq -A_1P^2 + B_1'P - C_1 \\ \leq -A_0P^2 + B_0'P - C_0 \end{aligned} \tag{4.13}$$

for all P. We may now interpret geometrically the bounds which have been determined for m. Plotting each member of (4.13) against P we obtain four parabolas, the loci of the quadratic forms, and one straight line through the origin corresponding to the term mP. We are interested in finding bounds on the unknown slope m of this line. By (4.13) we have that the line intersects none of the four parabolas. Now, conditions (4.2), (4.3) insure that the line does not intersect the two parabolas of (4.1). At the same time, conditions (4.6), (4.7) insure that the line does not intersect the parabolas of (4.5). How-

ever, by (4.13) it follows that if the line does not intersect the parabolas of (4.5) it will not intersect the parabolas of (4.1). This implies that the upper and lower bounds for the slope m of this line given in (4.6), (4.7) are better than (or at least as good as) the bounds of (4.2), (4.3). An analytic proof of this same result can be obtained directly from (4.13) but will be omitted here.

So far, starting with the initial set of admissible functions u_0, $\dot{u}_0$, v_0, $\dot{v}_0$ upper and lower bounds were obtained for m. Introducing the further functions u_1', v_1', a new set of admissible functions u_1, $\dot{u}_1$, v_1, $\dot{v}_1$ was constructed which furnished improved bounds. This process can be repeated indefinitely provided sequences of functions u_1', u_2', $\cdots$, and v_1', v_2', $\cdots$, can be found such that for $i = 1, 2, \cdots$,

$$u_i' = 0, \text{ on } \mathcal{C}, \qquad \Delta v_i' = 0, \text{ in } \mathcal{D},$$

and such that $H(u_i') \neq 0$, $H(v_i') \neq 0$. At the n'th step the set of admissible functions used is

$$\begin{aligned} u_n &= u_0 + \textstyle\sum_{i=1}^{n} g_i u_i', & \dot{u}_n &= \dot{u}_0 + \textstyle\sum_{i=1}^{n} h_i u_i', \\ v_n &= v_0 + \textstyle\sum_{i=1}^{n} k_i v_i', & \dot{v}_n &= \dot{v}_0 + \textstyle\sum_{i=1}^{n} l_i v_i', \end{aligned} \tag{4.14}$$

where

$$g_i = -\frac{H(u_{i-1}, u_i')}{H(u_i')}, \qquad h_i = -\frac{H(\dot{u}_{i-1}, u_i')}{H(u_i')},$$

$$l_i = -\frac{H(\dot{v}_{i-1}, v_i') - \frac{1}{2}\int_e \left(\log\frac{1}{r}\right)\frac{\partial v_i'}{\partial n}\,ds}{H(v_i')}, \tag{4.15}$$

$$k_i = -\frac{\frac{1}{2}\int_e f\frac{\partial v_i'}{\partial n}\,ds + H(v_{i-1}, v_i')}{H(v_i')},$$

$i = 1, \cdots, n$. The quantities of (3.38) when evaluated for the set of admissible functions of (4.14) yield the formulas

$$\begin{aligned} A_n &= A_0 + \sum_{i=1}^{n}\frac{[H(\dot{u}_{i-1}, u_i')]^2}{H(u_i')} \\ &\quad + \sum_{i=1}^{n}\frac{\left[H(\dot{v}_{i-1}, v_i') + \frac{1}{2}\int_e \left(\log\frac{1}{r}\right)\frac{\partial v_i'}{\partial n}\,ds\right]^2}{H(v_i')}, \\ B_n &= B_0 + 2\sum_{i=1}^{n}\frac{H(u_{i-1}, u_i')\cdot H(\dot{u}_{i-1}, u_i')}{H(u_i')}, \\ C_n &= C_0 + \sum_{i=1}^{n}\frac{[H(u_{i-1}, u_i')]^2}{H(u_i')} + \sum_{i=1}^{n}\frac{\left[\frac{1}{2}\int_e f\frac{\partial v_i'}{\partial n}\,ds - H(v_{i-1}, v_i')\right]^2}{H(v_i')}, \\ B_n' &= B_0' - 2\frac{\left[H(v_{i-1}, v_i') + \frac{1}{2}\int_e f\frac{\partial v_i'}{\partial n}\,ds\right]\left[H(\dot{v}_{i-1}, v_i') - \frac{1}{2}\int_e \left(\log\frac{1}{r}\right)\frac{\partial v_i'}{\partial n}\,ds\right]}{H(v_i')}. \end{aligned} \tag{4.16}$$

Our final result is now stated in the following theorem which follows by mathematical induction from Theorem 3 and the results of this section.

Theorem 4. For each value of $n = 0, 1, 2, \cdots$,

$$
\begin{aligned}
B_n - 2\sqrt{A_nC_n} &\leq 2\pi w(x_0, y_0) \leq B_n + 2\sqrt{A_nC_n} \\
B'_n - 2\sqrt{A_nC_n} &\leq 2\pi w(x_0, y_0) \leq B'_n + 2\sqrt{A_nC_n}
\end{aligned} \tag{4.17}
$$

where A_n, B_n, C_n, B'_n, $n = 1, 2, \cdots$, are the quantities of (4.16) and A_0, B_0, C_0, B'_0 are the quantites of (3.38) evaluated for the initial set of functions $u_0, \dot{u}_0, v_0, \dot{v}_0$. In addition,

$$
\begin{aligned}
B_{n+1} + 2\sqrt{A_{n+1}C_{n+1}} &\leq B_n + 2\sqrt{A_nC_n}, \\
B'_{n+1} + 2\sqrt{A_{n+1}C_{n+1}} &\leq B'_n + 2\sqrt{A_nC_n}, \\
B_n - 2\sqrt{A_nC_n} &\leq B_{n+1} - 2\sqrt{A_{n+1}C_{n+1}}, \\
B'_n - 2\sqrt{A_nC_n} &\leq B'_{n+1} - 2\sqrt{A_{n+1}C_{n+1}},
\end{aligned}
$$

so that at each iteration, $n = 1, 2, \cdots$, the upper bounds will decrease (or remain constant) while the lower bounds will increase (or remain constant).

This theorem provides us with a definite procedure for obtaining successively improved upper and lower bounds for the solution w of the Dirichlet problem I at any point (x_0, y_0) of $\mathcal{D}$.

5. Inequalities for the derivatives. The three-dimensional problem. The method we have used can be adapted to furnish upper and lower bounds for the derivatives of w in $\mathcal{D}$. The inequalities $L_1 \leq W_1 \leq U_1$ of (2.3) are utilized without change; however, we must replace $L_2 \leq W_2 \leq U_2$ and $L_3 \leq W_3 \leq U_3$ by new inequalities $L'_2 \leq W'_2 \leq U'_2$ and $L'_3 \leq W'_3 \leq U'_3$. These new inequalities are obtained by applying (2.2) of Theorem 1 to the boundary value problems

$$
\text{II}' \quad \begin{cases} \Delta w' = 0, \text{ in } \mathcal{D}, \\ w' = -\vartheta(P \log 1/r), \text{ on } \mathcal{C}, \end{cases}
$$

and

$$
\text{III}' \quad \begin{cases} \Delta \omega' = 0, \text{ in } \mathcal{D}, \\ \omega' = f - \vartheta(P \log 1/r), \text{ on } \mathcal{C}, \end{cases}
$$

which take the place of the problems II and III previously considered. Here, ϑ is the differential operator being considered, derivatives being taken with respect to ξ and η where $r = [(x - \xi)^2 + (y - \eta)^2]^{\frac{1}{2}}$, and where, after differentiation, ξ is to be replaced by x_0 and η by y_0.

For example, we shall consider the case where we are interested in getting upper and lower bounds for the value of $\partial w/\partial x$ at the point (x_0, y_0). The operator ϑ is chosen to be $\partial/\partial\xi$ where, after differentiation, (ξ, η) is to be replaced by (x_0, y_0). The derivation of the inequalities $L'_i \leq W'_i \leq U'_i$, $i = 2, 3$, follows exactly the same steps as the derivation of $L_i \leq W_i \leq U_i$, $i = 2, 3$, in Section 3, and will not be repeated here. These inequalities can be combined with $L_1 \leq$

$W_1 \leq U_1$ as before, to yield inequalities directly upon $2\pi P \partial w/\partial x \,|_{(x_0, y_0)}$. Finally, by eliminating P we can determine $\partial w/\partial x \,|_{(x_0, y_0)}$ to lie in the intersection of two intervals, as was previously done for $w(x_0, y_0)$.

It is clear that these considerations can be carried through, inasmuch as they depend only upon

1. The fundamental inequalities (2.2) of Theorem 1. These remain unchanged.

2. The representation theorem (3.4) for a function $\overline{w} \in C^{(2)}$. By differentiating this formula, we obtain the corresponding one for $2\pi \partial \overline{w}/\partial x \,|_{(x_0, y_0)}$ which is to be used in an exactly analogous manner to the use of (3.4) in Section 3.

3. Formula (3.22) giving the solution of I in terms of the Green's function. This is to be replaced now by the expression obtained by differentiating both sides of (3.22).

4. The linearity of the problem.

Let us turn now to the Dirichlet problem in 3-space. We find that the considerations of the preceding sections carry over at once to furnish upper and lower bounds for the solution. The changes to be made are the obvious ones. In place of $\log 1/r$ in the auxiliary problem II, one must use the fundamental solution, $1/r$, of Laplace's equation in 3-space. In place of (3.4) we use the representation theorem in 3-space, and in place of (3.22) we use Green's formula written for 3-space.

6. The Poisson equation. With slight changes one can adapt the method to find upper and lower bounds for the solution of the following boundary-value problem associated with the Poisson equation:

$$\text{(i)} \qquad \Delta w = -p, \text{ in } \mathcal{D}, \qquad w = 0, \text{ on } \mathcal{C},$$

where $p(x, y)$ is a continuous function of x and y in $\mathcal{D}$, and the assumptions on $\mathcal{D}$ and $\mathcal{C}$ are the same as in Section 1. We shall treat the plane problem, although the theorems are extensible to 3-space.

To apply the method, it is necessary to obtain the fundamental inequalities for (i) and also for the problems

$$\text{(ii)} \qquad \Delta \overline{w} = 0, \text{ in } \mathcal{D}, \qquad \overline{w} = -P \log 1/r, \text{ on } \mathcal{C},$$

and

$$\text{(iii)} \qquad \Delta \omega = -p, \text{ in } \mathcal{D}, \qquad \omega = -P \log 1/r, \text{ on } \mathcal{C},$$

where P is a constant. We note that neither (i) nor (iii) conform to the Dirichlet type of problem I, whereas (ii) is identical with problem II previously considered. Thus, we can not directly apply Theorem 1. In fact, it is more convenient to begin afresh and derive the complementary variational principles which apply specifically to problem (iii). We are led to a set of fundamental inequalities for (iii). Specializing these by taking $p = 0$ we get inequalities for (ii); alternatively by taking $P = 0$ we get inequalities for (i).

These inequalities can be combined to yield upper and lower bounds for the solution of (i) at a preassigned point (x_0, y_0) of (i) in a form similar to Theorem 2. Eliminating P as before, the analogue of Theorem 3 is found. An iteration procedure is established as in Section 4. The final results we summarize as follows. Beginning with an initial set of admissible functions $u_0, \dot{u}_0, v_0, \dot{v}_0$ such that[6]

$$\begin{aligned} &u_0 = 0, \text{ on } \mathcal{C}, \ \dot{u}_0 = -\log 1/r, \text{ on } \mathcal{C}, \\ &\Delta v_0 = -p, \text{ in } \mathfrak{D}, \ \Delta \dot{v}_0 = 0, \text{ in } \mathfrak{D}, \end{aligned} \tag{6.1}$$

and two sequences of functions $u_1', u_2', \cdots, v_1', v_2', \cdots$, satisfying the conditions

$$u_i' = 0, \text{ on } \mathcal{C}, \qquad \Delta v_i' = 0, \text{ in } \mathfrak{D},$$

$H(u_i') \neq 0, H(v_i') \neq 0, i = 1, 2, \cdots$, we construct four sequences of functions

$$\begin{aligned} u_n &= u_0 + \textstyle\sum_{i=1}^{n} g_i u_i', \qquad v_n = v_0 + \textstyle\sum_{i=1}^{n} k_i v_i' \\ \dot{u}_n &= \dot{u}_0 + \textstyle\sum_{i=1}^{n} h_i u_i', \qquad \dot{v}_n = \dot{v}_0 + \textstyle\sum_{i=1}^{n} l_i v_i' \end{aligned} \tag{6.2}$$

$n = 1, 2, \cdots$, where

$$\begin{aligned} g_i &= -\frac{H(u_{i-1}, u_i') - \frac{1}{2}\int_{\mathfrak{D}} p u_i' \, dA}{H(u_i')}, \\ h_i &= -\frac{H(\dot{u}_{i-1}, u_i')}{H(u_i')}, \qquad k_i = -\frac{H(v_{i-1}, v_i')}{H(v_i')} \\ l_i &= \frac{H(\dot{v}_{i-1}, v_i') - \frac{1}{2}\int_{\mathcal{C}} \left(\log \frac{1}{r}\right)\left(\frac{\partial v_i'}{\partial n}\right) ds}{H(v_i')}, \end{aligned} \tag{6.3}$$

We are led to the formulas

$$\begin{aligned} A_n &= A_0 + \sum_{i=1}^{n} \frac{[H(\dot{u}_{i-1}, u_i')]^2}{H(u_i')} + \sum_{i=1}^{n} \frac{\left[H(\dot{v}_{i-1}, v_i') - \frac{1}{2}\int_{\mathcal{C}} \left(\log \frac{1}{r}\right)\frac{\partial v_i'}{\partial n} ds\right]^2}{H(v_i')} \\ B_n &= B_0 + 2\sum_{i=1}^{n} \frac{H(\dot{u}_{i-1}, u_i')\left[H(u_{i-1}, u_i') - \frac{1}{2}\int_{\mathfrak{D}} p u_i' \, dA\right]}{H(u_i')}, \\ C_n &= C_0 + \sum_{i=1}^{n} \frac{[H(v_{i-1}, v_i')]^2}{H(v_i')} + \sum_{i=1}^{n} \frac{\left[H(u_{i-1}, u_i') - \frac{1}{2}\int_{\mathfrak{D}} p u_i' \, dA\right]^2}{H(u_i')} \\ B_n' &= B_0' - 2\sum_{i=1}^{n} \frac{H(v_{i-1}, v_i')\left[H(\dot{v}_{i-1}, v_i') - \frac{1}{2}\int_{\mathcal{C}} \left(\log \frac{1}{r}\right)\frac{\partial v_i'}{\partial n} ds\right]}{H(v_i')}, \end{aligned} \tag{6.4}$$

[6] These conditions as well as the notations of this section are not to be confused with those of the preceding sections.

where $n = 1, 2, 3, \cdots$, and

$$A_0 = -H(\dot{u}_0) - H(\dot{v}_0) + \int_c \left(\log \frac{1}{r}\right) \frac{\partial \dot{v}_0}{\partial n}\, ds,$$

$$B_0 = -2H(u_0, \dot{u}_0) + \int_{\mathcal{D}} p(\dot{u}_0 + \log 1/r)\, dA,$$

(6.5)

$$C_0 = -H(u_0) + \int_{\mathcal{D}} pu_0\, dA - H(v_0).$$

$$B_0' = 2H(v_0, \dot{v}_0) - \int_c \left(\log \frac{1}{r}\right) \frac{\partial v_0}{\partial n}\, ds + \int_D p \log \frac{1}{r}\, dA.$$

The following theorem is completely analogous to Theorem 4 of Section 4.

Theorem 5. For each value of $n = 0, 1, 2, \cdots$,

$$B_n - 2\sqrt{A_n C_n} \leq 2\pi w(x_0, y_0) \leq B_n + 2\sqrt{A_n C_n},$$

(6.6)

$$B_n' - 2\sqrt{A_n C_n} \leq 2\pi w(x_0, y_0) \leq B_n' + 2\sqrt{A_n C_n},$$

where A_n, B_n, C_n, B_n' are defined by (6.4) and (6.5). In addition, at each successive iteration $n = 1, 2, \cdots$, the upper bounds in (6.6) decrease (or remain constant) while the lower bounds increase (or remain constant).

This theorem provides us with a definite iteration procedure for obtaining successively refined upper and lower bounds for the solution of the boundary value problem (i) at any point (x_0, y_0) of $\mathcal{D}$. We shall make use of this procedure in the next section in working out an example. It should be noticed that due to the manner in which the iteration process is defined, it is necessary neither to solve systems of linear equations nor, what is equivalent, to introduce sequences of orthonormal functions to obtain successive improvement in the upper and lower bounds.

7. Example. We now proceed to apply the results of the preceding section to obtain the solution of the following specific problem. Let $\mathcal{D}$ be the square domain in the xy-plane with vertices at $(1, -1)$, $(1, 1)$, $(-1, 1)$, $(-1, -1)$ and let $\mathcal{C}$ denote the boundary of this square. The boundary-value problem we shall consider is this

$$\Delta w = -1, \text{ in } \mathcal{D}, \qquad w = 0, \text{ on } \mathcal{C}, \tag{7.1}$$

where $\mathcal{D}$ and $\mathcal{C}$ now refer to the specific square domain under consideration. This is a problem of the type (i) considered in Section 6. In particular, we shall apply the method of inequalities, as given in Section 6 to determine upper and lower bounds for the quantity $w(0, 0)$, which is the value of the solution w of (7.1), at the origin $(0, 0)$.

Several physical interpretations may be placed upon this problem. In the field of elasticity, for example, this problem corresponds to that of determining the lateral deflection w at the center of a square, homogeneous membrane, with fixed edges, subjected to a uniform tension along the edges and a uniform lateral

pressure (assuming a proper choice of units). Similarly, we may interpret w as the stress function which occurs in the study of the torsion of a prismatic bar with square cross-section; the analogy existing in elasticity between the torsion problem and the deflection of a stretched membrane being of course, a familiar one.[7] Several interesting hydrodynamical interpretations of w are also possible.[8] The primary purpose of the example we have selected, however, is to provide an illustration of the method; determination of the value $w(0, 0)$, in itself, constitutes nothing new, since the solution of (7.1) for the domain under consideration can be obtained explicitly in series form.

The first step is to select an initial set of admissible functions u_0, $\dot{u}_0$, v_0, $\dot{v}_0$. We choose these functions as follows, the domain of definition in each case, is the square, $\mathfrak{D}$:

$$u_0 \equiv 0, \tag{7.2}$$

$$\dot{u}_0 \equiv \begin{cases} -\tfrac{1}{2}r^2 + 2r - 3/2, & r \leq 1, \\ -\log 1/r, & r \geq 1, \end{cases} \tag{7.3}$$

$$v_0 \equiv -\frac{1}{4}r^2, \tag{7.4}$$

$$\dot{v}_0 \equiv 0, \tag{7.5}$$

where $r = [x^2 + y^2]^{\frac{1}{2}}$ is the distance measured from (0, 0) the point under consideration. The function $\dot{u}_0$, in (7.3), is exactly the truncated function of (3.32) with $P = 1$ and $R = 1$.

We proceed to evaluate A_0, B_0, C_0, B_0' for the functions we have chosen, using formulas (6.5) in which p is now equal to 1. We tabulate the values of the various non-vanishing integrals which occur:

$$H(\dot{u}_0) = \pi\left(\frac{11}{12} + \log_e 2\right) - 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}, \tag{7.6}$$

$$\int_{\mathfrak{D}} (\dot{u}_0 + \log 1/r)\, dA = \pi/12, \tag{7.7}$$

$$H(v_0) = 1/3, \tag{7.8}$$

$$-\int_c (\log 1/r)\frac{\partial v_0}{\partial n}\, ds + \int_{\mathfrak{D}} \log 1/r\, dA = 2. \tag{7.9}$$

In evaluating (7.9), we observe, by Green's first identity, that the sum of the two integrals is just equal to $2H(\log 1/r, v_0)$, which has the value 2. We digress, to observe that for numerical purposes we must replace the series

$$S = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}, \tag{7.10}$$

[7] Love, A. E. H., *A treatise on the mathematical theory of elasticity*, 4th edition, Cambridge, 1934, p. 322.

[8] Ibid., p. 314, under (b), (c).

occurring in (7.6), by an approximate value obtained by summing the series to a finite number of terms and dropping the remainder; the exact value of S appears to be difficult to obtain. We shall use the value

$$1 - \frac{1}{3^2} + \frac{1}{5^2} - + \cdots - \frac{1}{43^2} = .91584. \tag{7.11}$$

Since (7.10) is an alternating series, this value is too low; the maximum error, however, is less than $\frac{1}{45^2} = .00049$. By taking the value too low we ensure that our final inequalities will not be violated because of the error incurred by this approximation. Returning now to our computation, we find that

$$A_0 = -3.2257, \qquad B_0 = 0.26180, \qquad C_0 = 0.33333, \qquad B_0' = 2.$$

Applying (6.6) of Theorem 5 we have

$$-1.8120 \leq 2\pi w(0, 0) \leq 2.3356, \qquad -0.0738 \leq 2\pi w(0, 0) \leq 4.0738$$

and hence

$$-.0738 \leq 2\pi w(0, 0) \leq 2.3356 \tag{7.13}$$

The interval (7.13) constitutes our *zero-order estimate* for $w(0, 0)$.

Examining (7.13), we observe that the lower bound has no practical value, since à priori we know that $w(0, 0)$ is actually a positive number. The zero-order estimate is, therefore, unsatisfactory, and so we proceed to a first-order estimate, using the iteration procedure of Section 6.

For the function u_1', we choose

$$u_1' \equiv (x^2 - 1)(y^2 - 1). \tag{7.14}$$

For the function v_1', we choose the real part of the analytic function z^4, where $z = x + iy$, and $i = \sqrt{-1}$; thus,

$$v_1' \equiv x^4 - 6x^2y^2 + y^4. \tag{7.15}$$

We now compute A_1, B_1, C_1, B_1', from formulas (6.4). We tabulate the values of the various non-vanishing integrals whose values are required:

$$H(u_0, u_1') = -\pi 101/60, \tag{7.16}$$

$$H(u_1') = 2^7/5 \cdot 3^2 \tag{7.17}$$

$$\int_c (\log 1/r)(\partial v_1'/\partial n)\, ds = 16\pi - (160/3), \tag{7.18}$$

$$\int_\Omega u_1'\, dA = 2^4/3^2, \tag{7.19}$$

$$H(v_0, v_1') = 2^4/3 \cdot 5 \tag{7.20}$$

$$H(v_1') = 2^3 \cdot 3/5 \cdot 7 \tag{7.21}$$

Using these values, and the values of A_0, B_0, C_0, B_0' previously obtained, we find that

$$A_1 = -.65786,\ B_1 = 1.9144,\ C_1 = -.00375,\ B_1' = 1.85086. \tag{7.22}$$

Applying (6.6) of Theorem 5 we have

$$1.8151 \leq 2\pi w(0, 0) \leq 2.0137, \qquad 1.7515 \leq 2\pi w(0, 0) \leq 1.9502$$

and hence

$$1.8171 \leq 2\pi w(0, 0) \leq 1.9502. \tag{7.23}$$

The interval (7.23) constitutes our *first-order estimate* for $w(0, 0)$.

We observe that the first-order estimate has served to bracket $w(0, 0)$ between two positive values. Computing the ratio of the difference of these two values to the smaller of the two, we find that the maximum error which could be incurred by taking $w(0, 0)$ to be any value in the interval (7.23), is approximately 7.1%

We carry the iteration procedure one step further to obtain a second-order estimate for $w(0, 0)$. For the function u_2' which is required, we choose

$$u_2' \equiv (x^4 - 1)(x^4 - 1). \tag{7.24}$$

For the function v_2', we choose the real part of the analytic function z^8; thus

$$v_2' \equiv x^8 - 28x^6y + 70x^4y^4 - 28x^2y^6 + y^8. \tag{7.25}$$

We proceed to compute A_2, B_2, C_2, B_2' from formulas (6.4). To do this we must determine the functions u_1, $\dot{u}_1$, v_1, $\dot{v}_1$, of (6.2). These functions are completely determined once we have the values of the constants g_1, h_1, k_1, l_1. We compute g_1, h_1, k_1, l_1, from the formulas (6.3). We find using the values of the integrals involved (note that these have already been computed) that

$$g_1 = \frac{5}{2^4}, \quad h_1 = \pi\frac{3(101)}{2^{10}}, \quad k_1 = -\frac{7}{2^4\cdot 3^2}, \quad l_1 = \left(16\pi - \frac{160}{3}\right)\left(\frac{5\cdot 7}{2^9\cdot 3}\right), \tag{7.26}$$

We tabulate the values of the integrals, not already listed, which enter into our computations of A_2, B_2, C_2, and C_2':

$$H(u_2') = \frac{2^{11}}{3^2\cdot 5\cdot 7}, \qquad H(u_1', u_2') = \frac{2^{11}}{3\cdot 5^2\cdot 7}, \qquad H(\dot{u}_0, u_2') = -\frac{1708\pi}{2^7\cdot 3\cdot 5},$$

$$\int_D u_2'\,dA = \frac{2^6}{5^2}, \quad H(v_0, v_2') = -\frac{2^7}{3^2\cdot 5}, \quad \int_c \left(\log\frac{1}{r}\right)\frac{\partial v_2'}{\partial n}\,ds = 2^6\left(4\pi - 11 - \frac{51}{35}\right),$$

$$H(v_2') = \frac{2^9\cdot 4237}{5\cdot 9\cdot 11\cdot 13}, \qquad H(v_1', v_2') = -\frac{2^{11}\cdot 13}{3^2\cdot 5\cdot 11}.$$

Carrying through all the substitutions and the numerical computations which remain we arrive at the values

$$A_2 = -.58869,\ B_2 = 1.9018,\ C_2 = -.00302,\ B_2' = 1.85122. \tag{7.28}$$

Applying (6.6) of Theorem 5, we have

$$1.8174 \leq 2\pi w(0, 0) \leq 1.9862, \qquad 1.7669 \leq 2\pi w(0, 0) \leq 1.9356,$$

and hence

$$1.8174 \leq 2\pi w(0, 0) \leq 1.9356. \tag{7.29}$$

The interval (7.29) constitutes our *second-order estimate* for $w(0, 0)$. Carrying through the division by 2π and retaining only four figures as being significant, we find that

$$.2892 \leq w(0, 0) \leq .3081 \tag{7.30}$$

Thus, $w(0, 0)$ can be no larger than .3081 and *no smaller than* .2892. For practical purposes we might wish to take any intermediate value as the actual value for $w(0, 0)$. The maximum error in any such approximation, is, of course, determined and is approximately 6.5%. Taking for $w(0, 0)$ the arithmetic mean of the upper and lower bounds in (7.30) would give a value in error by at most some 3.3%.

We remark that replacing our square domain $\mathfrak{D}$ by one of length $2a$ on each side, and replacing the constant 1, in (7.1), by a constant p, has the effect of multiplying the upper and lower bounds given in (7.30) by $\frac{p}{a^2}$.

It is interesting to compare the estimates we have obtained for $w(0, 0)$ with the known value of the solution. The solution of (7.1) is given by the series[9]

$$w(x, y) = \frac{16}{\pi^3} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^3} (-1)^{(n-1)/2} \left(1 - \frac{\cosh n\pi y/2}{\cosh n\pi/2}\right) \cos \frac{n\pi x}{2} \tag{7.31}$$

Evaluating this at (0, 0) we find that

$$w(0, 0) = \frac{16}{\pi^3} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^3} (-1)^{(n-1)/2} \left(1 - \frac{1}{\cosh n\pi/2}\right). \tag{7.32}$$

Taking the sum of the first three terms of this alternating series, we find that

$$w(0, 0) = .29574 \tag{7.33}$$

with a maximum error of $-.00151$. The value (7.33) lies within the predicted interval (7.30).

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[9] Timoshenko, S., *Theory of elasticity*, New York, 1934, p. 246.

DIFFERENCE AND ASSOCIATED OPERATORS, WITH SOME APPLICATIONS

By W. G. Bickley

1. Introduction. The object of this paper is to present a comprehensive, consistent, and logical scheme of difference and associated operators, and its kernel is the Table II on p. 186.

Many computers have their attention so closely focussed upon the actual differences which they produce and/or use, that they do not always take kindly to the idea of a difference 'operator'. Logically, however, an operation can be distinguished both from the function upon which it operates and from the result which it produces, and the symbolic expression of such discrimination is practically of considerable value—as the universal use made of the differential operator D testifies.

Certain difference operators are in common use, and for some there is an accepted notation. For others agreement is not general, and in any case there appear to be gaps in the scheme.

2. Notation and Definitions. We are concerned with a function $f(x)$ of an independent variable x, and with its values at a sequence of equidistant values of x, and denote by w the interval in x. If x_0 is a convenient pivotal value of x, we shall frequently (for brevity) denote $x_0 + nw$ by x_n and $f(x_0 + nw) = f(x_n)$, by f_n.

We use five fundamental operators:—

(i) E, defined by

$$Ef_n = f_{n+1};$$

(ii) Δ, the forward difference operator, defined by

$$\Delta f_n = f_{n+1} - f_n;$$

(iii) δ, the central difference operator, defined by

$$\delta f_n = f_{n+\frac{1}{2}} - f_{n-\frac{1}{2}};$$

(iv) ∇, the backward difference operator, defined by

$$\nabla f_n = f_n - f_{n-1};$$

(v) D, the differential operator, defined by

$$Df_n = (df/dx)_{x=x_n}$$

We use also

(vi) μ, the averaging operator, defined by

$$\mu f_n = \tfrac{1}{2}(f_{n+\frac{1}{2}} + f_{n-\frac{1}{2}})$$

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and also operators inverse to four of the fundamental five,

(vii) Σ, the forward summation operator, such that

$$\nabla\Sigma \equiv 1 \quad \text{or} \quad \Sigma f_n = \Sigma f_{n-1} + f_n$$

(viii) σ, the central summation operator, such that

$$\delta\sigma \equiv 1 \quad \text{or} \quad \sigma f_n = \sigma f_{n-1} + f_{n-\frac{1}{2}}$$

(ix) $\mathfrak{Z}$, the backward summation operator, such that

$$\Delta\mathfrak{Z} \equiv 1 \quad \text{or} \quad \mathfrak{Z} f_n = \mathfrak{Z} f_{n-1} + f_{n-1}$$

(x) I, the integral operator, such that

$$DI \equiv 1 \quad \text{or} \quad If_n = \int^{x_n} f(x)\,dx.$$

Each of the four inverse operations implies an arbitrary additive constant, whose value is made determinate in any specific case by an additional condition, usually some 'initial' condition, inherent in the problem under discussion.

All the operations may be repeated, and repetitions will be indicated by arabic numeral indices. The 'powers' of Σ, σ, $\mathfrak{Z}$, or I are indeterminate to the extent of a polynomial in n or x of degree one less than the index.

It is customary to omit the function symbol, and attach the suffix indicating the value of the argument to the operator, e.g., to write δ^2_r for $\delta^2 f_r$; this will be done where no confusion can arise.

3. Sum and Difference Table. In forming a sum and difference table from a function tabulated at equidistant values of the argument, there is no ambiguity (apart from the additive constants just mentioned) about the entries, but in any derived column the same quantity may be denoted by any of three different symbols. This is shown in Table I, where the entries in any space are different modes of symbolising the *same* quantity.

TABLE I

Sum and Difference Table

	$\mathfrak{Z}_{-1}\ \sigma_{-\frac{1}{2}}\ \Sigma_{-1}$	f_{-2}		
$\mathfrak{Z}_0^2\ \sigma_{-1}^2\ \Sigma_{-2}^2$			$\Delta_{-2}\ \delta_{-\frac{1}{2}}\ \nabla_{-1}$	
	$\mathfrak{Z}_0\ \sigma_{-\frac{1}{2}}\ \Sigma_{-1}$	f_{-1}		$\Delta_{-2}^2\ \delta_{-1}^2\ \nabla_0^2$
$\mathfrak{Z}_1^2\ \sigma_0^2\ \Sigma_{-1}^2$			$\Delta_{-1}\ \delta_{-\frac{1}{2}}\ \nabla_0$	
	$\mathfrak{Z}_1\ \sigma_{\frac{1}{2}}\ \Sigma_0$	f_0		$\Delta_{-1}^2\ \delta_0^2\ \nabla_1^2$
$\mathfrak{Z}_2^2\ \sigma_1^2\ \Sigma_0^2$			$\Delta_0\ \delta_{\frac{1}{2}}\ \nabla_1$	
	$\mathfrak{Z}_2\ \sigma_{\frac{3}{2}}\ \Sigma_1$	f_1		$\Delta_0^2\ \delta_1^2\ \nabla_2^2$
		f_2	$\Delta_1\ \delta_{\frac{3}{2}}\ \nabla_2$	

The above indicates the notation employed; the three entries in any compartment are different symbols representing the same quantity.

4. Relations between Operators. In general, the above operators obey the laws of algebra. All the operators are expressible in terms of any one of the fundamental five. Important well-known instances of these are

$$E = 1 + \Delta, \quad \delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}, \quad E = e^{wD}$$

of which the last is the symbolic form of Taylor's theorem. A complete set of these formulae is given in Table II.

For analytical purposes—the speedy and systematic derivation of important formulae—another operator is highly convenient, namely

$$\vartheta = \tfrac{1}{2}wD$$

In terms of this,

$$\delta = 2 \sinh \vartheta, \quad \mu = \cosh \vartheta, \quad E = e^{2\vartheta}$$

Formulae in terms of ϑ are also given in Table II.

A number of relations also exist between three or more operators. The complete set is too numerous to give, and we content ourselves with a short list of the more useful ones.

$$\Delta = E\nabla \qquad \nabla = E^{-1}\Delta$$

$$E^{-\frac{1}{2}}\Delta = \delta = E^{\frac{1}{2}}\nabla, \qquad \Delta\nabla = \delta^2$$

$$\mu\delta = \tfrac{1}{2}(E - E^{-1}) = \tfrac{1}{2}(\Delta + \nabla) = \sinh wD = \sinh 2\vartheta$$

$$E^{\frac{1}{2}} = \mu + \tfrac{1}{2}\delta \qquad E^{-\frac{1}{2}} = \mu - \tfrac{1}{2}\delta$$

$$\mu^2 = 1 + \tfrac{1}{4}\delta^2$$

5. Applications of the formulae. These formulae are of great assistance in reproducing or developing formulae for the various processes in numerical mathematics, e.g., interpolation, differentiation, integration. The symbols obey algebraic laws, and may be freely used; even the purist will, if wise, accept them as a means of exploration, furnishing a posteriori justifications when he feels the need to do so. We give a few samples illustrating the technique.

(i) Newton-Gregory interpolation formula.

$$f(x_0 + pw) = E^p f_0 = (1 + \Delta)^p f_0$$

$$= \left(1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \cdots\right) f_0$$

$$= f_0 + p\Delta_0 + \frac{p(p-1)}{2}\Delta_0^2 + \cdots$$

(ii) Everett's interpolation formula.

With $q = 1 - p$, we have

$$E^p = \frac{E^{p+1} - E^{-q}}{E - E^{-1}} = \frac{E^{p+1} - E^{1-p} + E^q - E^{-q}}{E - E^{-1}}$$

$$= \frac{E \cdot 2 \sinh 2p\vartheta + \sinh 2q\vartheta}{2 \sinh 2\vartheta} = \frac{\sinh 2q\vartheta}{\sinh 2\vartheta} + \frac{\sinh 2p\vartheta}{\sinh 2\vartheta} E.$$

TABLE II

	E	Δ	δ	∇	D	ϑ
E	E	$1+\Delta$	$1+\frac{1}{2}\delta^2+\delta\sqrt{(1+\frac{1}{4}\delta^2)}$	$1/(1-\nabla)$	e^{wD}	$e^{2\vartheta}$
Δ	$E-1$	Δ	$\delta\sqrt{(1+\frac{1}{4}\delta^2)}+\frac{1}{2}\delta^2$	$\nabla/(1-\nabla)$	$e^{wD}-1$	$e^{2\vartheta}-1$
δ	$E^{\frac{1}{2}}-E^{-\frac{1}{2}}$	$\Delta(1+\Delta)^{-\frac{1}{2}}$	δ	$\nabla(1-\nabla)^{-\frac{1}{2}}$	$2\sinh\frac{1}{2}wD$	$2\sinh\vartheta$
∇	$1-E^{-1}$	$\Delta/(1+\Delta)$	$\delta\sqrt{(1+\frac{1}{4}\delta^2)}-\frac{1}{2}\delta^2$	∇	$1-e^{-wD}$	$1-e^{-2\vartheta}$
D	$\frac{1}{w}\log_e E$	$\frac{1}{w}\log(1+\Delta)$	$\frac{2}{w}\sinh^{-1}\frac{1}{2}\delta$	$\frac{1}{w}\log\{1/(1-\nabla)\}$	D	$2\vartheta/w$
μ	$\frac{1}{2}(E^{\frac{1}{2}}+E^{-\frac{1}{2}})$	$(1+\frac{1}{2}\Delta)(1+\Delta)^{-\frac{1}{2}}$	$\sqrt{(1+\frac{1}{4}\delta^2)}$	$(1-\frac{1}{2}\nabla)(1-\nabla)^{-\frac{1}{2}}$	$\cosh\frac{1}{2}wD$	$\cosh\vartheta$
Σ	$E/(E-1)$	$(1+\Delta)/\Delta$	$\sqrt{(1+\frac{1}{4}\delta^2)}/\delta+\frac{1}{2}$	∇^{-1}	$(1-e^{-wD})^{-1}$	$(1-e^{-2\vartheta})^{-1}$
σ	$E^{\frac{1}{2}}/(E^{\frac{1}{2}}-1)$	$(1+\Delta)^{\frac{1}{2}}/\Delta$	δ^{-1}	$(1-\nabla)^{\frac{1}{2}}/\nabla$	$\frac{1}{2}\operatorname{cosech}\frac{1}{2}wD$	$\frac{1}{2}\operatorname{cosech}\vartheta$
$\mathfrak{Z}$	$1/(E^{\frac{1}{2}}-1)$	Δ^{-1}	$\sqrt{(1+\frac{1}{4}\delta^2)}/\delta-\frac{1}{2}$	$(1-\nabla)/\nabla$	$(e^{wD}-1)^{-1}$	$(e^{2\vartheta}-1)^{-1}$
I	$w/\log E$	$w/\log(1+\Delta)$	$w/2\sinh^{-1}\frac{1}{2}\delta$	$w/\log\{1/(1-\nabla)\}$	D^{-1}	$w/2\vartheta$

Using the fact that $\sinh \vartheta = \frac{1}{2}\delta$, and the known expansions for $\sinh 2q\vartheta/\cosh \vartheta$ and $\sinh 2p\vartheta/\cosh \vartheta$, we derive

$$f_p = E^p f_0$$

$$= \left\{q + \frac{(q+1)q(q-1)}{3!}\delta^2 + \frac{(q+2)(q+1)q(q-1)(q-2)}{5!}\delta^4 + \cdots\right\} f_0$$

$$+ \left\{p + \frac{(p+1)p(p-1)}{3!}\delta^2 + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\delta^4 \cdots\right\} f_1$$

which is, effectively, Everett's formula.

(iii) Numerical differentiation. Forward difference formulae.

The result

$$wD = \log(1 + \Delta)$$

gives the formula for the first derivative in terms of forward differences,

$$wf_0' = wD\, f_0 = (\Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{3}\Delta^3 - \cdots) f_0$$

Higher derivatives are found in terms of powers of the series on the right, i.e.

$$w^r f_0^{(r)} = (wD)^r f_0 = (\Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{3}\Delta^3 - \cdots)^r f_0$$

Apart from slow convergence, this formula has the disadvantage that the leading term $(\Delta^r f_0)$ is on a line considerably below f_0 in the difference table, unless r is quite small. Formulae for $(wD)^{2s} E^{-s} f_0$ and $(wD)^{2s+1} E^{-s} f_0$ would be preferable, and are readily obtainable, and their coefficients have properties simpler than those of the series obtained by expansion of the formula last written. They are, however, inferior to central difference formulae.

Formulae in terms of ∇, corresponding to the above, are readily written down.

(iv) Numerical differentiation. Central difference formulae.

The formula

$$wD = 2 \sinh^{-1} \tfrac{1}{2}\delta$$

is not immediately available for the first (or any odd) derivative at tabular points, because the expansion contains only odd powers of δ, and these do not correspond to entries in the difference table if f has an integer suffix. For even derivatives, the formula will serve, and we can conveniently write it

$$w^{2s} f_0^{(2s)} = (wD)^{2s} f_0 = \left(\frac{2}{\delta} \sinh^{-1} \frac{\delta}{2}\right)^{2s} \cdot \delta^{2s} f_0 .$$

For odd derivatives, we must use the averages of the odd differences below and above the central line, i.e., we must introduce the averager μ, and derive

$$w^{2s+1} f_0^{(2s+1)} = (wD)^{2s+1} f_0 = \left\{\frac{1}{\mu}\left(\frac{2}{\delta} \sinh^{-1} \frac{\delta}{2}\right)^{2s+1}\right\} \mu\delta^{2s+1} f_0 .$$

For points at half-interval, however, the formula for odd indices,

$$w^{2s+1}f_{\frac{1}{2}}^{(2s+1)} = (wD)^{2s+1}f_{\frac{1}{2}} = \left(\frac{2}{\delta}\sinh^{-1}\frac{\delta}{2}\right)^{2s+1}\cdot\delta^{2s+1}f_{\frac{1}{2}}$$

is available, using entries direct from the table. For even indices we use

$$w^{2s}f_{\frac{1}{2}}^{(2s)} = (wD)^{2s}f_{\frac{1}{2}} = \left\{\frac{1}{\mu}\left(\frac{2}{\delta}\sinh^{-1}\frac{\delta}{2}\right)^{2s}\right\}\mu\delta^{2s}f_{\frac{1}{2}}.$$

Clearly the early coefficients in the series for

$$\left(\frac{2}{\delta}\sinh^{-1}\frac{\delta}{2}\right)^{r} \quad \text{and} \quad \left\{\frac{1}{\mu}\left(\frac{2}{\delta}\sinh^{-1}\frac{\delta}{2}\right)^{r}\right\}$$

will be useful, and a number of these have been worked out by the present writer, and may be published, along with other sets of coefficients, later.

Formally, however,

$$\left(\frac{2}{\delta}\sinh^{-1}\frac{\delta}{2}\right)^{r} = 1 - \frac{r}{24}\delta^2 + \frac{5r^2+22r}{5760}\delta^4 - \frac{35r^3+462r^2+1528r}{29\,03040}\delta^6$$
$$+ \frac{175r^4+4620r^3+40724r^2+1\,19856r}{13934\,59200}\delta^8$$
$$- \frac{385r^5+16940r^4+2\,79884r^3+20\,57968r^2+56\,82048r}{36\,78732\,28800}\delta^{10}\cdots$$
$$= 1 + \sum_{m=1}^{\infty}(-)^m rC_m(r)\delta^{2m}$$

and

$$\frac{1}{\mu}\left(\frac{2}{\delta}\sinh^{-1}\frac{\delta}{2}\right)^{r} = 1 - \frac{r+3}{24}\delta^2 + \frac{5r^2+52r+135}{5760}\delta^4$$
$$- \frac{35r^3+777r^2+5749r+14175}{29\,03040}\delta^6$$
$$+ \frac{175r^4+6720r^3+96794r^2+6\,19776r+14\,88375}{13934\,59200}\delta^8$$
$$- \frac{385r^5+22715r^4+5\,36294r^3+63\,33250r^2+374\,08281r+884\,09475}{36\,78732\,28800}\delta^{10}\cdots$$
$$= 1 + \sum_{m=1}^{\infty}(-)^m(r+2m+1)C_m(r+1)\delta^{2m}$$

(v) Numerical integration: forward and backward difference formulae.

$$\int_0^w f(x_0+z)\,dz = w\int_0^1 f(x_0+pw)\,dp$$
$$= w\int_0^1 (1+\Delta)^p\,dp\cdot f_0 = w\frac{\Delta}{\log(1+\Delta)}f_0$$
$$= w(1+\tfrac{1}{2}\Delta - \tfrac{1}{12}\Delta^2 + \tfrac{1}{24}\Delta^3 - \tfrac{19}{720}\Delta^4\cdots)f_0$$

Also
$$\int_0^w f(x_0 + z)\,dz = w\int_0^1 (1-\nabla)^{-p}\,dp\cdot f_0$$
$$= w\,\frac{\nabla}{(1-\nabla)\log\{1/(1-\nabla)\}} f_0$$
$$= w(1 + \tfrac{1}{2}\nabla + \tfrac{5}{12}\nabla^2 + \tfrac{3}{8}\nabla^3 + \tfrac{251}{720}\nabla^4 \cdots) f_0$$

This last is a formula for forward integration, upon which is based the Adams-Bashforth method for the numerical solution of first order differential equations. It has the disadvantage of slow convergence, and also needs to be supplemented by an à posteriori check. Such a check is provided by the result

$$\int_0^w f(x_0 + z)dz = E\int_w^0 f(x_0 + z)\,dz$$
$$= w\int_{-1}^0 (1-\nabla)^{-p}\,dp\cdot Ef_0 = w\,\frac{\nabla}{\log\{1/(1-\nabla)\}} f_1$$
$$= w(1 - \tfrac{1}{2}\nabla - \tfrac{1}{12}\nabla^2 - \tfrac{1}{24}\nabla^3 - \tfrac{19}{720}\nabla^4 \cdots) f_1$$

The above formulae may be more expeditiously obtained by use of the operator $I = D^{-1}$, for

$$\int_0^w f(x_0 + z)\,dz = If_1 - If_0 = \Delta If_0$$
$$= \Delta D^{-1} f_0 = w\,\frac{\Delta}{\log(1+\Delta)} f_0$$

and similarly for the others.

Formulae for repeated integrals may be constructed by use of powers of the series in the above, but are for most purposes inferior to the central difference formulae to be given later.

The Gregory formula for an integral over a longer range is, however, important. Essentially it corrects the trapezoidal rule, and we denote the 'trapezoidal sum' by T_n, defined by

$$T_n = \tfrac{1}{2}f_0 + f_1 + f_2 + \cdots + f_{n-1} + \tfrac{1}{2}f_n$$
$$= \frac{1}{2}\,\frac{E+1}{E-1}(E^n - 1)f_0$$
$$= \frac{1}{2}\left\{\frac{1 - \tfrac{1}{2}\nabla}{\nabla} f_n - \frac{1 + \tfrac{1}{2}\Delta}{\Delta} f_0\right\}$$

Also

$$\int_0^{nw} f(x_0 + z)\,dz = w\int_0^n E^p\,dp\cdot f_0$$
$$= \frac{E^n - 1}{\log E} f_0 = \frac{1}{\log E}(f_n - f_0)$$
$$= \frac{1}{\log\{1/(1-\nabla)\}} f_n - \frac{1}{\log(1+\Delta)} f_0$$

Hence

$$\int_0^{nw} f(x_0+z)\,dz - wT_n = \left\{\frac{1+\frac{1}{2}\Delta}{\Delta} - \frac{1}{\log(1+\Delta)}\right\} f_0 - \left\{\frac{1-\frac{1}{2}\nabla}{\nabla} - \frac{1}{\log\{1/(1-\nabla)\}}\right\} f_n$$

$$= (\tfrac{1}{12}\Delta - \tfrac{1}{24}\Delta^2 + \tfrac{19}{720}\Delta^3 \cdots) f_0 - (\tfrac{1}{12}\nabla + \tfrac{1}{24}\nabla^2 + \tfrac{19}{720}\nabla^3 \cdots) f_n$$

which is, effectively, Gregory's formula.

The Euler-Maclaurin summation formula yields to a somewhat similar technique, using expressions in terms of D or ϑ.

(vi) Numerical integration: central difference formulae. We have

$$\int_0^w f(x_0+z)\,dz = \int_0^1 e^{2p\vartheta} w\,dp\cdot f_0 = w\,\frac{\sinh\vartheta}{\vartheta}\,e^{\vartheta} f_0 = w\left\{\frac{1}{\mu}\left(\frac{2}{\delta}\sinh^{-1}\frac{\delta}{2}\right)^{-1}\right\}\mu f_{\frac{1}{2}}$$

Using the series previously given, with $r = -1$, we deduce that

$$\int_0^w f(x_0+z)\,dz = w(1 - \tfrac{1}{12}\delta^2 + \tfrac{11}{720}\delta^4 \cdots)\mu f_{\frac{1}{2}}$$

of which the right-hand side may be written in the more familiar form

$$w\{\tfrac{1}{2}(f_0+f_1) - \tfrac{1}{24}(\delta_0^2+\delta_1^2) + \tfrac{11}{1440}(\delta_0^4+\delta_1^4) \cdots\}$$

The formula may be obtained more expeditiously by more intensive use of the operational technique,

$$\int_0^w f(x_0+z)z\,d = \delta I f_{\frac{1}{2}} = \frac{\delta}{D} f_{\frac{1}{2}} = \left\{\frac{1}{\mu}\left(\frac{2}{\delta}\sinh^{-1}\frac{\delta}{2}\right)^{-1}\right\}\mu f_{\frac{1}{2}}$$

Formulae for repeated integrals may be obtained, and in use those of odd order will be more convenient if the averager μ is artificially incorporated.

In the numerical solution of differential equations we endeavour to build up progressive values of an indefinite integral If, I^2f, $\cdots$, made definite by initial conditions. One method of considerable practical use consists in calculating from the differential equation an appropriate difference of this integral—and it will usually be most convenient if the order of the difference is the same as the order of the integral. Again, for odd orders, we artificially introduce the averager μ.

Symbolically

$$\delta^{2s} I^{2s} f_0 = w^{2s}\left(\frac{2}{\delta}\sinh^{-1}\frac{\delta}{2}\right)^{-2s} f_0$$

and

$$\delta^{2s+1} I^{2s+1} f_{\frac{1}{2}} = w^{2s+1}\left\{\frac{1}{\mu}\left(\frac{2}{\delta}\sinh^{-1}\frac{\delta}{2}\right)^{-(2s+1)}\right\}\mu f_{\frac{1}{2}}$$

Additional to the formula for $\delta If_{\frac{1}{2}}$ given above we quote

$$\delta^2 I^2 f_0 = w^2\{1 + \tfrac{1}{12}\delta^2 - \tfrac{1}{240}\delta^4 + \tfrac{31}{60480}\delta^6 \cdots\}f_0$$

$$\delta^3 I^3 f_{\frac{1}{2}} = w^3\{1 + \tfrac{1}{24}\delta^2 - \tfrac{31}{5760}\delta^4 \cdots\}\mu f_{\frac{1}{2}}$$

$$\delta^4 I^4 f_0 = w^4\{1 + \tfrac{1}{6}\delta^2 - \tfrac{7}{120}\delta^4 + \tfrac{31}{3024}\delta^6 \cdots\}f_0$$

(Further coefficients, and coefficients for higher orders, have been worked out.)

(vii) Sums and integrals.

A method for the numerical integration of differential equations, having some advantages over the method implicit in the last section, consists in converting sums into integrals by means of a correction term involving differences of the integrand (summand). Formulae depending on forward, central, or backward sums and differences are accessible, but we concentrate here upon those involving central sums and differences, because these are normally so much more convenient in use. Again, we shall need the averager in connection with odd orders, so we consider first the simpler case of even orders.

We have

$$I^{2s} = w^{2s}\left(\frac{1}{2\sinh^{-1}\frac{\delta}{2}}\right)^{2s} = \frac{w^{2s}}{\delta^{2s}}\left(\frac{2}{\delta}\sinh^{-1}\frac{\delta}{2}\right)^{-2s}$$

$$= w^{2s}\{\sigma^{2s} + \textstyle\sum_{m=1}^{s}(-)^{m+1}2sC_m(-2s)\sigma^{2s-2m} + \sum_{m=s+1}^{\infty}(-)^{m+1}2sC_m(-2s)\delta^{2m-2s}\}$$

Similarly

$$I^{2s+1} = w^{2s+1}\left(\frac{1}{2\sinh^{-1}\frac{\delta}{2}}\right)^{2s+1} = \frac{w^{2s+1}}{\delta^{2s+1}}\left\{\frac{1}{\mu}\left(\frac{2}{\delta}\sinh^{-1}\frac{\delta}{2}\right)^{-(2s+1)}\right\}\cdot\mu$$

$$= w^{2s+1}\{1 + \textstyle\sum_{m=1}^{s-1}(-)^{m+1}2(s-m)C_m(-2s)\sigma^{2s+1-2m} + \sum_{m=s+1}^{\infty}(-)^{m}2(m-s)C_m(-2s)\delta^{2m-2s-1}\}\mu$$

As particular instances of these formulae we quote

$$If_0 = w\{\tfrac{1}{2}(\sigma_{\frac{1}{2}} + \sigma_{-\frac{1}{2}}) - \tfrac{1}{24}(\delta_{\frac{1}{2}} + \delta_{-\frac{1}{2}}) + \tfrac{1}{1440}(\delta_{\frac{1}{2}}^3 + \delta_{-\frac{1}{2}}^3) - \tfrac{19}{967680}(\delta_{\frac{1}{2}}^5 + \delta_{-\frac{1}{2}}^5)\cdots\}$$

$$I^2 f_0 = w^2\{\sigma_0^2 + \tfrac{1}{12}f_0 - \tfrac{1}{240}\delta_0^2 + \tfrac{31}{60480}\delta_0^4 \cdots\}$$

$$I^3 f_0 = w^3\{\tfrac{1}{2}(\sigma_1^3 + \sigma_{-1}^3) + \tfrac{1}{240}(\delta_1 + \delta_{-1}) - \tfrac{31}{60480}(\delta_1^3 + \delta_{-1}^3)\cdots\}$$

$$I^4 f_0 = w^4\{\sigma_0^4 + \tfrac{1}{6}\sigma_0^2 - \tfrac{7}{120}f_0 + \tfrac{31}{3024}\delta_0^2 \cdots\}$$

(viii) Some additional results.

Central difference formulae possess two marked advantages over forward or backward difference formulae, namely, greater rapidity of convergence, and the occurrence of alternate orders of differences (or sums) only. The latter is, how-

ever, largely discounted when odd orders are in question, at pivotal points. This last feature may be removed by constructing formulae for intervals which are even multiples of w—a device which, as we shall see, possesses other merits.

As a first, and possibly the most important, instance, we consider an integral over an interval $2w$. (The result must, of course, represent the finite difference form of Simpson's rule.)

$$\int_{-w}^{w} f(x_0 + z)\,dz = \int_{-1}^{1} e^{\frac{1}{2}p\vartheta} w\,dp \cdot f_0$$

$$= w\,\frac{\sinh 2\vartheta}{\vartheta} f_0 = w\,\frac{\delta\sqrt{(1+\frac{1}{4}\delta^2)}}{\sinh^{-1}\frac{\delta}{2}} f_0$$

$$= 2w\{1 + \tfrac{1}{6}\delta^2 - \tfrac{1}{180}\delta^4 + \tfrac{1}{1512}\delta^6 - \tfrac{23}{43200}\delta^8 \cdots\} f_0$$

(It is the first two terms of this series which are equivalent to Simpson's rule, and the third gives the well-known 'error' term, $w\delta^4 f_0/90$.)

As used for forward integration, to determine f_1, this demands an estimate of $\delta^2 f_0$. Although it is usually possible, after the process is well under way, to make a sufficiently accurate estimate, there is virtue in a formula which can use a *known* $\delta^2 f_0$, needing estimates only of higher differences; this suggests using a range of $4w$ for the integral.

We have

$$\int_{-2w}^{2w} f(x_0 + z)\,dz = w\,\frac{\sinh 4\vartheta}{\vartheta} f_0 = w\,\frac{2\delta(1+\frac{1}{2}\delta^2)\sqrt{(1+\frac{1}{4}\delta^2)}}{\sinh^{-1}\frac{\delta}{2}} f_0$$

$$= 4w\{1 + \tfrac{2}{3}\delta^2 + \tfrac{7}{90}\delta^4 - \tfrac{2}{945}\delta^6 + \tfrac{13}{37800}\delta^8 \cdots\} f_0$$

(The first two terms of this series are equivalent to Milne's formula for the integral, $4w(2f_{-1} - f_0 + 2f_1)/3$.)

Examples could readily be multiplied, but enough have been given to show the success of the operational technique in rapidly developing useful formulae, and, it is hoped, the combined logical and practical advantages to be derived from thinking and writing operationally.

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UPPER AND LOWER BOUNDS FOR THE SOLUTION OF THE FIRST BIHARMONIC BOUNDARY VALUE PROBLEM[1]

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In this paper[2] we consider the boundary value problem

$$\begin{aligned} \Delta\Delta w &= p, && \text{in } R, \\ w &= f; \quad \partial w/\partial n = g, && \text{on } C, \end{aligned} \tag{1}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, R is a plane domain with boundary C, $\partial/\partial n$ denotes differentiation in the direction of the outer normal to C, and p, f, and g are given functions. The solution w of (1) admits of various physical interpretations, e.g., as the deflection of a thin elastic plate occupying the plane domain R, with prescribed deflection f and slope g along the boundary, subjected to a distributed transverse load, of intensity pD, D being the flexural rigidity of the plate.

It is well known that under suitable restrictions on p, f, g, and R, the boundary value problem (1) possesses one and only one solution. Assuming the existence and uniqueness of the solution of (1), we propose to describe a method for obtaining upper and lower bounds for $w(x_0, y_0)$, the value of w at a point (x_0, y_0) of R, the point (x_0, y_0) being given in advance. The method is applicable to many other problems, and may be used to obtain upper and lower bounds for the derivatives of w at (x_0, y_0); but, for definiteness, these considerations will be excluded here.

The first section contains an elementary inequality which is of use in establishing the inequalities for $w(x_0, y_0)$ desired in the second section. These inequalities constitute the main result of the present paper. The third section deals with a familiar procedure for successively improving the upper and lower bounds. The paper concludes with a few remarks concerning the applicability of the method and a comparison with another approach already existing in the literature.

1. An inequality. We shall employ the classical Green's identity

$$\int_R \Delta\phi\Delta\psi \, dx\, dy = \begin{cases} \displaystyle\int_R \phi\Delta\Delta\psi \, dx\, dy + \int_C \left(\Delta\psi \frac{\partial\phi}{\partial n} - \phi \frac{\partial\Delta\psi}{\partial n}\right) ds, \\ \displaystyle\int_R \psi\Delta\Delta\phi \, dx\, dy + \int_C \left(\Delta\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\Delta\phi}{\partial n}\right) ds, \end{cases} \tag{2}$$

and Schwarz' inequality

$$[H(\phi, \psi)]^2 \leq H(\phi) \cdot H(\psi), \tag{3}$$

[1] This paper contains the results presented to the American Mathematical Society on February 22, 1947 and April 26, 1947.

[2] This paper was written for the Watertown Arsenal under a contract in Applied Mechanics. The authors are indebted to the authorities of Watertown Arsenal for the release of this paper for publication.

where

$$H(\phi, \psi) = \int_R \Delta\phi \cdot \Delta\psi \, dx \, dy; \qquad H(\phi) = \int_R (\Delta\phi)^2 \, dx \, dy. \tag{3'}$$

For the sake of uniformity, it will be convenient to write $\Delta\Delta w$, w, and $\partial w/\partial n$, instead of p, f, and g, respectively, in our formulas. But it must be kept in mind that $\Delta\Delta w$ is a known function in R, and that w and $\partial w/\partial n$ are known functions on C.

Let u be a function satisfying the boundary conditions of (1), i.e.

$$u = w, \qquad \partial u/\partial n = \partial w/\partial n, \qquad \text{on } C; \tag{4}$$

and let v be a function satisfying the partial differential equation of (1), i.e.

$$\Delta\Delta v = \Delta\Delta w \qquad \text{in } R. \tag{5}$$

It follows that (see (3'))

$$\left.\begin{aligned} H(u - w) \\ H(v - w) \end{aligned}\right\} \leqq H(u - v). \tag{6}$$

The inequality (6) is easily verified by first observing that

$$H(u - v) = H((u - w) + (w - v)) = H(u - w) + H(w - v), \tag{7}$$

where use has been made of the fact that

$$H(u - w, w - v) = 0$$

by Green's identity.

2. Upper and lower bounds for the value of the solution at a given point. Let (x_0, y_0) be a given point of R, which we shall consider as fixed throughout the discussion. The bi-Laplacian $\Delta\Delta w$ is a given function in R, and w and $\partial w/\partial n$ are given functions on C. In this section we shall obtain upper and lower bounds for the number $w(x_0, y_0)$.

Together with the function w of (1), it will be convenient to consider another function, $\overline{w}$, the solution of the following boundary value problem:

$$\begin{aligned} \Delta\Delta\overline{w} &= 0 && \text{in } R, \\ \overline{w} &= -r^2 \log r, \qquad \partial\overline{w}/\partial n = -\partial(r^2 \log r)/\partial n, && \text{on } C. \end{aligned} \tag{8}$$

The function $\overline{w}$ is the regular function associated with the Green's function g with pole at (x_0, y_0), which is defined by

$$g(x, y; x_0, y_0) = r^2 \log r + \overline{w}(x, y).$$

In analogy with the functions u and v associated with the w of (1), we shall also consider functions $\bar{u}$ and $\bar{v}$ associated with the $\overline{w}$ of (8). The function $\bar{u}$ satisfies

$$\bar{u} = \overline{w}, \qquad \partial\bar{u}/\partial n = \partial\overline{w}/\partial n, \qquad \text{on } C, \tag{9}$$

while $\bar{v}$ satisfies

$$\Delta\Delta\bar{v} = \Delta\Delta\bar{w} = 0, \qquad \text{in } R. \tag{10}$$

We recall that from Green's identity[3] we have

$$\begin{aligned} 8\pi w(x_0, y_0) &= \int_R (r^2 \log r)\Delta\Delta w \, dx \, dy \\ &+ \int_C \left(w \frac{\partial \Delta(r^2 \log r)}{\partial n} - \Delta(r^2 \log r) \frac{\partial w}{\partial n} \right) ds \\ &+ \int_C \left(\Delta w \frac{\partial (r^2 \log r)}{\partial n} - r^2 \log r \frac{\partial \Delta w}{\partial n} \right) ds. \end{aligned} \tag{11}$$

From this last equation it is clear that in order to obtain upper and lower bounds for $w(x_0, y_0)$ it is only necessary to obtain upper and lower bounds for the last line integral of (11), which is the only integral not depending entirely on known functions.

We are now ready to derive two sets of upper and lower bounds for $w(x_0, y_0)$. From Schwarz' inequality we have that

$$[H(u - w, \bar{u} - \bar{w})]^2 \leqq H(u - w) \cdot H(\bar{u} - \bar{w}). \tag{12}$$

Using the inequality (6) of the last section, (12) becomes

$$[H(u - w, \bar{u} - \bar{w})]^2 \leqq H(u - v) \cdot H(\bar{u} - \bar{v}). \tag{13}$$

But, from Green's identity (see (2)) we have

$$\begin{aligned} H(u - w, \bar{u} - \bar{w}) &= H(u, \bar{u}) - H(\bar{u}, w) - H(u - w, \bar{w}) \\ &= H(u, \bar{u}) - \int_R \bar{u}\Delta\Delta w \, dx \, dy \\ &- \int_C [\Delta w \, \partial\bar{u}/\partial n - \bar{u} \, \partial\Delta w/\partial n] \, ds. \end{aligned} \tag{14}$$

The last line integral appearing in (14) is nothing but the "unknown" line integral of (11). Adding and subtracting the two other integrals of the right hand side of (11) to the right hand side of (14), and combining the resulting equation with (13), we obtain

$$(8\pi w(x_0, y_0) - b)^2 \leqq ac, \tag{15}$$

where

$$\begin{aligned} a &= H(u - v), \qquad c = H(\bar{u} - \bar{v}), \\ b &= -H(u, \bar{u}) + \int_R \bar{u}\Delta\Delta w \, dx \, dy \\ &+ \int_R (r^2 \log r)\Delta\Delta w \, dx \, dy \\ &+ \int_C \left(w \frac{\partial(\Delta r^2 \log r)}{\partial n} - \Delta(r^2 \log r) \frac{\partial w}{\partial n} \right. ds. \end{aligned} \tag{16}$$

[3] See, for instance, Frank-Mises, "Die Differential-und Integralgleichungen der Mechanik und Physik", 2nd edition, vol. 1, p. 858.

Given four functions u, v, $\bar{u}$, and $\bar{v}$, satisfying conditions (4), (5), (9), and (10), respectively, equations (15) and (16) yield upper and lower bounds for $w(x_0, y_0)$.

Another set of bounds can be obtained by starting with

$$[H(v - w, \bar{v} - \overline{w})]^2 \leqq H(v - w) \cdot H(\bar{v} - \overline{w}), \tag{17}$$

instead of (12). From (6) it then follows that

$$[H(v - w, \bar{v} - \overline{w})]^2 \leqq H(u - v) \cdot H(\bar{u} - \bar{v}), \tag{18}$$

and from (2) it follows that

$$\begin{aligned} H(v - w, \bar{v} - \overline{w}) &= H(v, \bar{v}) + H(\overline{w}, w - v) - H(w, \bar{v}) \\ &= H(v, \bar{v}) + \int_c \left(\Delta(w - v)\frac{\partial \overline{w}}{\partial n} - \overline{w}\frac{\partial \Delta(w - v)}{\partial n}\right) ds \\ &\quad - \int_c \left(\Delta\bar{v}\frac{\partial w}{\partial n} - w\frac{\partial \Delta\bar{v}}{\partial n}\right) ds. \end{aligned} \tag{19}$$

The "unknown" line integral of (11) appears, with a minus sign, on the right hand side of (19). Adding and subtracting the two other integrals of the right hand side of (11) to the right hand side of (19), and combining the resulting equation with (18), we obtain

$$(8\pi w(x_0, y_0) - b')^2 \leqq ac, \tag{20}$$

where

$$\begin{aligned} a &= H(u - v), \qquad c = H(\bar{u} - \bar{v}), \\ b' &= H(v, \bar{v}) - \int_c \left(\Delta v\frac{\partial \overline{w}}{\partial n} - \overline{w}\frac{\partial \Delta v}{\partial n}\right) ds \\ &\quad - \int_c \left(\Delta\bar{v}\frac{\partial w}{\partial n} - w\frac{\partial \Delta\bar{v}}{\partial n}\right) ds + \int_R (r^2 \log r)\Delta\Delta w \, dx \, dy \\ &\quad + \int_c \left(w\frac{\partial(\Delta r^2 \log r)}{\partial n} - \Delta(r^2 \log r)\frac{\partial w}{\partial n}\right) ds. \end{aligned} \tag{21}$$

Again, given four functions u, v, $\bar{u}$, and $\bar{v}$, satisfying conditions (4), (5), (9), and (10), respectively, equations (20) and (21) yield upper and lower bounds for $w(x_0, y_0)$. The relations (15), (16), (20) and (21) are the inequalities we wished to derive. It is to be noticed that in both (15) and (20) the "error", which is measured by ac, is the same. However, the "approximation" b depends only on u and $\bar{u}$, the functions required to satisfy boundary conditions alone, while b' depends only on v and $\bar{v}$, functions required to satisfy the partial differential equations alone.

3. An iteration procedure. In order to improve the bounds given by (15) and (20) one may add to u a finite linear combination of functions u_i satisfying the homogeneous boundary conditions

$$u_i = \partial u_i/\partial n = 0, \qquad \text{on } C, \tag{22}$$

add to v a finite linear combination of functions v_i satisfying the homogeneous differential equation

$$\Delta\Delta v_i = 0, \qquad \text{in } R, \tag{23}$$

and then minimize $H(u - v)$ in order to determine the coefficients of the best linear combinations. That this gives the best approximation follows from (16) and (21). This would lead to two sets of linear equations for the coefficients. However, in order to obtain explicit formulas for the nth iteration, we shall find it convenient to require the functions u_i and v_i to be orthonormalized in advance as follows:

$$\int_R \Delta u_i \Delta u_j \, dx \, dy = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases} \tag{24}$$

and

$$\int_R \Delta v_i \Delta v_j \, dx \, dy = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases} \tag{25}$$

In numerical applications this means that, instead of solving two sets of linear equations, we shall orthonormalize the functions u_i and v_i in advance. Then the resulting two sets of linear equations will have matrices of coefficients of diagonal form, and will be solvable by inspection, which enables us to obtain explicit formulas for the nth iteration. The numerical labor is thus shifted from that of solving two sets of linear equations to the equivalent task of orthonormalizing the functions u_i and v_i in advance.

Before going further, it is worthwhile to notice that for any functions u and v (see section 1) we have, by Green's identity

$$\begin{aligned} &H(u - v) = H((u - w) + (w - v)) = H(u - w) + H(v - w), \\ &H(u - w, u_i) = H(u - v, u_i) + H(v - w, u_i) = H(u - v, u_i), \\ &H(v - w, v_i) = H(v - u, v_i) + H(u - w, v_i) = H(v - u, v_i). \end{aligned} \tag{26}$$

A similar set of equations is obtained by replacing u, v, w in (26) by $\bar{u}$, $\bar{v}$, $\bar{w}$ respectively.

Let us now replace u and v in $H(u - v)$ by $u + \sum_{i=1}^{n} \alpha_i u_i$ and $v + \sum_{i=1}^{n} \beta_i v_i$ respectively, and determine the optimum values of the real numbers α_i and β_i. From (26) it follows that

$$\begin{aligned} &H(u + \textstyle\sum_{i=1}^{n} \alpha_i u_i - v - \sum_{i=1}^{n} \beta_i v_i) \\ &\qquad = H(u - w + \textstyle\sum_{i=1}^{n} \alpha_i u_i) + H(v - w + \sum_{i=1}^{n} \alpha_i v_i). \end{aligned} \tag{27}$$

The desired optimum values of the α_i and β_i are then seen to be the "Fourier coefficients"

$$\begin{aligned} \alpha_i &= -H(u - w, u_i) = -H(u - v, u_i), \\ \beta_i &= -H(v - w, v_i) = H(u - v, v_i), \end{aligned} \tag{28}$$

the last equalities having been obtained from (26). In a similar manner, by starting with $H(\bar{u} - \bar{v})$ we are led to consider the Fourier coefficients

$$\begin{aligned} \tilde{\alpha}_i &= -H(\bar{u} - \bar{w}, u_i) = -H(\bar{u} - \bar{v}, u_i) = -H(\bar{u}, u_i) \\ \tilde{\beta}_i &= -H(\bar{v} - \bar{w}, v_i) = H(\bar{u} - \bar{v}, v_i). \end{aligned} \tag{29}$$

Let u, v, $\bar{u}$, $\bar{v}$ be the functions originally started with. At the nth step in the iteration, u and v will be replaced by $u + \sum_{i=1}^{n}\alpha_i u_i$ and $v + \sum_{i=1}^{n}\beta_i v_i$ respectively (α_i and β_i given by (28)), and $\bar{u}$ and $\bar{v}$ will be replaced by $\bar{u} + \sum_{i=1}^{n}\tilde{\alpha}_i u_i$ and $\bar{v} + \sum_{i=1}^{n}\tilde{\beta}_i v_i$ respectively ($\tilde{\alpha}_i$ and $\tilde{\beta}_i$ given by (29)). Since the α_i, β_i, $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ are known explicitly, we may compute from (16) and (21) the numbers a_n, c_n, b_n and b_n' corresponding to the nth step. The final result, obtained with the use of (26), (27), (28), (29) is

$$\begin{aligned} a_n &= H(u - v) - \sum_{i=1}^{n}\alpha_i^2 - \sum_{i=1}^{n}\beta_i^2, \\ c_n &= H(\bar{u} - \bar{v}) - \sum_{i=1}^{n}\tilde{\alpha}_i^2 - \sum_{i=1}^{n}\tilde{\beta}_i^2, \\ b_n &= b + \sum_{i=1}^{n}\alpha_i\tilde{\alpha}_i, \qquad b_n' = b' - \sum_{i=1}^{n}\beta_i\tilde{\beta}_i. \end{aligned} \tag{30}$$

Thus, given the initial functions u, v, $\bar{u}$, $\bar{v}$ and the orthonormalized sequences of functions u_i and v_i, the computation of the successive approximations to $8\pi w(x_0, y_0)$ reduces to the evaluation of the Fourier coefficients (28) and (29), and then of the expressions (30).

If the sequences u_i and v_i are complete, in the sense specified below, then b_n and b_n' will both converge to $8\pi w(x_0, y_0)$. The sequence u_i is said to be complete if for any function u^* such that

$$u^* = \partial u^*/\partial n = 0, \quad \text{on } C, \tag{31}$$

we have

$$H(u^*) = \sum_{i=1}^{\infty}[H(u^*, u_i)]^2. \tag{32}$$

Similarly, the sequence v_i is said to be complete if for any function v^* such that

$$\Delta\Delta v^* = 0, \quad \text{in } R, \tag{33}$$

we have

$$H(v^*) = \sum_{i=1}^{\infty}[H(v^*, v_i)]^2. \tag{34}$$

Suppose that the sequences u_i and v_i are complete. From (26), (28), and (30) we may write

$$\begin{aligned} a_n = H(u - w) &- \sum_{i=1}^{n}[H(u - w, u_i)]^2 \\ &+ H(v - w) - \sum_{i=1}^{n}[H(v - w, v_i)]^2. \end{aligned} \tag{35}$$

Since $u - w$ satisfies (31), and $v - w$ satisfies (33), we have $\lim_{n\to\infty} a_n = 0$. A similar argument shows that $\lim_{n\to\infty} c_n = 0$. Therefore, from

$$(8\pi w(x_0, y_0) - b_n)^2 \leqq a_n c_n,$$

and

$$(8\pi w(x_0, y_0) - b_n')^2 \leqq a_n c_n,$$

it follows that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_n' = 8\pi w(x_0, y_0)$.

4. Concluding remarks. In order to apply the present results one has to choose four initial functions u, v, $\bar{u}$, and $\bar{v}$. In general, $\bar{u}$ is the most difficult to obtain. The function $-r^2 \log r$ cannot be used for $\bar{u}$ because $H(-r^2 \log r)$ diverges. However, one may always use for $\bar{u}$ a function coinciding with $-r^2 \log r$ outside a small circle with center at (x_0, y_0) and defined inside the circle to be an appropriate polynomial in r, for example. In some problems it is possible to construct functions u_i and v_i which satisfy (24) and (25) directly, but in most cases there is no other alternative than to start with a finite set of functions (satisfying the homogeneous boundary conditions or the homogeneous partial differential equation), and to orthonormalize them, employing the usual computational procedures. In addition, all of the definite integrals which enter into the computations must be evaluated with sufficient accuracy, so that the asserted inequalities for $w(x_0, y_0)$ are indeed inequalities. The amount of labor involved is not great provided the domain R and the boundary conditions are sufficiently simple so that it is possible to choose functions u, v, $\bar{u}$, $\bar{v}$, u_i and v_i for which the definite integrals involved can be evaluated explicitly in terms of elementary functions.

In his discussion of the convergence of Ritz' method, Trefftz,[4] has given approximations for the solution w of (1) (with $p \equiv 0$), at any point of the domain R. One approximation is in terms of functions satisfying the boundary conditions, and the other in terms of functions satisfying the differential equation. It is easily seen, however, that Trefftz' procedure does not furnish upper and lower bounds. Let $\epsilon \neq 0$ be a given real number and (x_0, y_0) be a given point of R. Then, according to Trefftz,

$$w(x_0, y_0) = \frac{1}{2\epsilon}[N(\epsilon) - N(-\epsilon)], \tag{36}$$

where $N(\pm\epsilon)$ is the minimum value of the functional

$$\tfrac{1}{2}H(u) \pm \epsilon\, u(x_0, y_0), \tag{37}$$

over the class of all functions u satisfying the boundary conditions

$$u = w = f, \quad \partial u/\partial n = \partial w/\partial n = g, \quad \text{on } C. \tag{38}$$

But it is clear that the "approximation"

$$\frac{1}{2\epsilon}[(\tfrac{1}{2}H(u) + \epsilon u(x_0, y_0) - (\tfrac{1}{2}H(u) - \epsilon u(x_0, y_0))], \tag{39}$$

obtained using a function u satisfying (38) is just $u(x_0, y_0)$ and hence may be

[4] *Konvergenz und Fehlerschätzung beim Ritzschen Verfahren*, Math. Annalen, vol. 100, 1928, p. 502 (especially pages 510 and 519).

either an upper or a lower bound for $w(x_0, y_0)$, depending upon the choice of the function u. A similar remark applies to Trefftz' approximation for $w(x_0, y_0)$ using biharmonic functions.

It may be mentioned that Trefftz' method for (1) is not applicable directly to the problem

$$\begin{aligned} \Delta w &= p, \qquad \text{in } R, \\ w &= f, \qquad \text{on } C, \end{aligned} \tag{40}$$

where p and f are given functions, because the Green's function for (40) is infinite at the pole. On the other hand, the procedure presented in this paper applies equally well to the problem (40).[5] One need only use $\int_R (\phi_x\psi_x + \phi_y\psi_y)\,dx\,dy$ instead of $\int_R \Delta\phi \cdot \Delta\psi\,dx\,dy$, and employ Green's formulas for Laplace's equation.

It is interesting to observe that from the inequalities (6) one may deduce immediately two known minimum principles associated with the solution w of (1). One principle isolates w from among the functions u satisfying the boundary conditions of (1) and the other picks out w from among the functions v satisfying the partial differential equation of (1).

Let us derive the first minimum principle. From (3)', $H(\phi) = 0$ if and only if ϕ is a harmonic function in R. Noting this fact, equations (6) and (7) imply that

$$H(v - w) \leqq H(u - v), \tag{41}$$

the equality sign holding if and only if $u = w$. Expanding both sides of (41), employing (2) and (4), yields

$$-2\int_R w\Delta\Delta v\,dx\,dy + H(w) \leqq -2\int_R u\Delta\Delta v\,dx\,dy + H(u). \tag{42}$$

In view of (3)' and (5), the result may be stated as follows: the functional

$$\int_R [(\Delta u)^2 - 2u\Delta\Delta w]\,dx\,dy, \tag{43}$$

is minimized, over the class of functions u satisfying the boundary conditions of (1), by the solution w of (1).

The second minimum principle may be derived similarly. Equations (6) and (7) imply that

$$H(u - w) \leqq H(u - v), \tag{44}$$

the equality sign holding if and only if $v - w$ is a harmonic function. With the aid of (2) and (5), equation (44) yields

[5] In this connection, compare the results obtained by H. J. Greenberg, *The Determination of Upper and Lower Bounds for the Solution of the Dirichlet Problem*, this Journal, 27, 161–182, (1948).

$$H(w) - 2\int_C (\Delta w\, \partial u/\partial n - u\, \partial \Delta w/\partial n)\, ds \leqq H(v) - 2\int_C (\Delta v\, \partial u/\partial n - u\, \partial \Delta v/\partial n)\, ds. \tag{45}$$

In view of (3)′ and (4), this result may be restated as follows: the functional

$$\int_R (\Delta v)^2\, dx\, dy - 2\int_C (\Delta v\, \partial w/\partial n - w\, \partial \Delta v/\partial n)\, ds, \tag{46}$$

is minimized, over the class of functions v satisfying the partial differential equation of (1), by any function $w + h$, where w is the solution of (1) and h is any function harmonic in R.[6]

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[6] This principle, for the case $w = \partial w/\partial n = 0$ on C, was formulated by U. Wegner, *Ein neues Verfahren zur Berechnung der Spannungen in Scheiben*, Forschungen auf dem Gebiete des Ingenieurwesens. Ausg. B, 13, 144–149, (1942).

THE INVERSE FUNCTIONS OF $z = w^{-1} \tanh w$ AND $z = w^{-1} \coth w$

By Abraham Hillman and Herbert E. Salzer

1. Theory of the Functions. The inverse functions $w(z)$ defined by the equations $z = w^{-1} \tanh w$ and $z = w^{-1} \coth w$ arise in the solution of the wave equation for rectangular and circular boundaries having finite impedance.[1]

In order to consider both equations simultaneously whenever possible, we let $F_1(w) = w^{-1} \tanh w$, $F_2(w) = w^{-1} \coth w$, and $w^{(i)}(z)$, $i = 1, 2$, stand for the inverse of $F_i(w)$. The i will be dropped when the statements are true for both values of i. Also note that $F(w) = w^{-1} \tanh (w + k)$ where $k = 0$ yields F_1 and $k = \pi i/2$ yields F_2. The k will be used in this sense in what follows.[2]

Since $F(w) = F(-w)$, $F(\overline{w}) = \overline{F(w)}$, and $w(\bar{z}) = \pm\overline{w(z)}$, where the bar is used to denote the complex conjugate, it suffices to take z in the upper half plane and w in the fourth quadrant.

Let us determine the singularities of $w(z)$. Since $F(w)$ is one-valued, its inverse $w(z)$ does not repeat any values and so cannot have any isolated essential singularities. $w(z)$ does not have any poles since $F(w)$ is not analytic at Infinity.

$w(z)$ has branch points at the z's corresponding to the multiple poles of $F(w)$ and the zeros of dF/dw. $F_1(w)$ has no multiple poles while $F_2(w)$ has a double pole at $w = 0$. This leads to a branch point of $w^{(2)}(z)$ at $z = \infty$. We now investigate the roots of $\dfrac{dF(w)}{dw} \equiv \dfrac{\operatorname{sech}^2(w + k)}{w} - \dfrac{\tanh(w + k)}{w^2} = 0$. This leads to $w = \sinh (w + k) \cosh (w + k)$ or $2w = \sinh 2(w + k)$.

When $k = \pi i/2$ this becomes $2w = -\sinh 2w$.

For both values of k, the equation has in the 4th quadrant a sequence of solutions, $W_n = R_n + iI_n$, $n = 0, 1, 2, \cdots$. It can be shown[3] that for the equation $2w = \sinh 2w$, $R_n - \frac{1}{2}\log [(4n + 1)\pi] \to 0$ and $-I_n - (n + \frac{1}{4})\pi \to 0$ as $n \to \infty$; while for $2w = -\sinh 2w$, $R_n - \frac{1}{2} \log [(4n - 1)\pi] \to 0$ and $-I_n - (n - \frac{1}{4})\pi \to 0$.

If we let $F(W_n)$, the n-th branch point in the upper half plane, be B_n, then

$$1 - B_n - B_n^2 W_n^2 = \operatorname{sech}^2(W_n + k) - \frac{\tanh (W_n + k)}{W_n} = 0 \text{ since } F'(W_n) = 0.$$

It follows from this and the fact that $W_n \to \infty$ as $n \to \infty$, that $B_n \to 0$ as $n \to \infty$.

Let $b_n (n = 0, 1, \cdots)$ be the segment from $z = 0$ to B_n. Let b_n' be its prolongation from B_n to infinity. Since $F''(W_n) \neq 0$, exactly two branches of $w(z)$ meet at each B_n. In what follows n, m, p and q are positive integers. The two branches of $w(z)$ meeting at B_n map b_n' on arcs in the w-plane joining W_n to points $-(q - \frac{1}{2})\pi i + k$. $(k = 0 \text{ or } \pi i/2)$.

Also, there are two arcs in the w-plane meeting at W_n on which $w(z)$ maps b_n.

[1] See Applied Mathematics Panel (NDRC-OSRD) Note. No. 18, June, 1945, which contains tables and graphs based on the material of this paper.

[2] Note that $i\, w^{(1)}(z)$ is the inverse of $(\tan w)/w$ and that $i\, w^{(2)}(-z)$ is the inverse of $(\cot w)/w$.

[3] See G. H. Hardy, Mess. of Math. N. S., 31, 1902, pp. 161–165.

One of these joins W_n to some point $-p\pi i + k$. The other goes from W_n to infinity with the line amp. of $w = -$amp. of B_n as an asymptote. This follows from the fact that for certain w's, $w(z)$ is approximately $1/z$ by virtue of the following theorem:

Theorem: Let $w = R + iI$ and $z = \rho \exp(i\phi)$ and let $k = ri$ with r real. Along any ray $\phi = c$ with $-\pi/2 < c < \pi/2$ there exists a continuous determination of $w(z)$ such that $w(z) - 1/z \to 0$ as $\rho \to 0$.

Proof:

$$w - \frac{1}{z} = w - \frac{w}{\tanh(w+k)} = w\left(1 - \frac{e^{w+k} + e^{-w-k}}{e^{w+k} - e^{-w-k}}\right) = \frac{-2we^{-2w-2k}}{1 - e^{-2w-2k}}$$

Hence

$$\left| w - \frac{1}{z} \right| = \frac{2\sqrt{R^2 + I^2}\, e^{-2R}}{\left| 1 - e^{-2R} e^{-2(I+r)i} \right|}$$

Now $| w - 1/z |$ can be made as small as desired by taking R large and positive while I is restricted by a condition such as $| I | \leqq e^{(2-\epsilon)R}$. This region in the w-plane is mapped by $w^{-1}\tanh(w + ri)$ (which is approximately $1/w$ for these w's) into a region in the z-plane which covers that part of every ray in the 1st and 4th quadrants for which ρ is sufficiently small. Q.E.D.

Since $F(w)$ is one-valued, no $w(b_n)$ or $w(b_n')$ has any points (except possibly end-points) in common with a $w(b_m)$ or $w(b_m')$ if $n \neq m$ while the four arcs, $w(b_n)$ and $w(b_n')$, mentioned above have only W_n in common. Topological considerations now show that for a given n, $p = n$ and $q = n$ and $n + 1$. Let $P_n(z)$ be the reciprocal power series for $w(z)$ having $w(\infty) = -(n - \frac{1}{2})\pi i + k$. The convergence of $P_n(z)$ is stopped by some B_m. $P_n(z)$ maps b_m' on an arc from W_m to $-(n - \frac{1}{2})\pi i + k$. From the previous paragraphs it follows that $m = n$ or $n - 1$. Since $| B_{n-1} | > | B_n |$, $P_n(z)$ converges for $| z | > | B_{n-1} |$. Similarly the Maclaurin expansion with initial term $-n\pi i + k$ converges for $| z | < | B_n |$. Now for every positive n, there exists a one-valued continuous determination of $w(z)$, defined on the simply connected region obtained by deleting from the complex plane b_{n-1}, b_n and their reflections with respect to the real axis, such that $w(\infty) = -(n - \frac{1}{2})\pi i + k$. This determination of $w(z)$ will be called the n-th branch, $w_n(z)$. The values of $w_n(z)$ on either side of the branch cut b_n are continuous with the values of $w_{n+1}(z)$ on the other side of b_n.

For the purpose of interpolation, it is only necessary to use values on one particular branch, with the exception of those z's very close to a b_n of the branch of interest. For such z's the values of the two branches [the n-th and (n + 1)-th] having b_n as common boundary may be needed.

Since there are no poles or isolated essential singularities, the only singularity besides the branch points, B_n, is $z = 0$, the limit of the B_n's. The above theorem on the approximation $1/z$ explains the behavior of $w(z)$ near the origin for those values of w not given by the various Maclaurin expansions.

For both functions $w(z)$ (involving $\tanh w$ and $\coth w$) the singularity B_0

is removed by using auxiliary functions. For the inverse of $(\tanh w)/w$, the expansion

$$[w_1(z)]^2 = -\pi^2/4 + c_1 z^{-1} + c_2 z^{-2} + \cdots$$

converging for $|z| > |B_1|$ is used. For the inverse of $(\coth w)/w$ the expansion

$$z^{\frac{1}{2}} w_1(z) = 1 + d_1 z^{-1} + d_2 z^{-2} + \cdots$$

also converging for $|z| > |B_1|$ (but a different B_1) is employed.

Of theoretical interest are expansions for $F(w)$ and $w(z)$ around the branch points. $F(w)$ in the neighborhood of W_n is given by

$$z = B_n + g_2(w - W_n)^2 + g_3(w - W_n)^3 + \cdots$$

where g_j is the value of $(d^jF/dw^j)/j!$ at $w = W_n$, since $F'(W_n) = 0$. Using the relations

$$\tanh(W_n + k) = W_n B_n \text{ and } \operatorname{sech}^2(W_n + k) = B_n$$

(which follows from $F(W_n) = B_n$ and $F'(W_n) = 0$), one finds[4] that $g_2 = -B_n^2$ and $g_3 = 2B_n/3W_n$. Then one finds by inverting the series $z - B_n = -B_n^2(w - W_n)^2 + (2B_n/3W_n)(w - W_n)^3 + \cdots$ that $w - W_n = (i/B_n)(z - B_n)^{1/2} - (1/3W_nB_n^2)(z - B_n) + \cdots$

The series for $w - W_n$ and $z - B_n$ have $|B_{n+1} - B_n|$ and $|-(n + \frac{1}{2})\pi i + k - W_n|$ respectively, as radii of convergence.

Numerical values of B_n and W_n for at least $n = 0, 1, 2, 3, 4$ are listed in attached tables.

The expansions when $n = 0$ can be obtained from the well-known Maclaurin expansions for $\tan x$ and $\cot x - 1/x$. For the function involving $\tanh w$, $B_0 = 1$, $W_0 = 0$, and one finds that

$$z - 1 = -\tfrac{1}{3}w^2 + \tfrac{2}{15}w^4 - \tfrac{17}{315}w^6 + \tfrac{62}{2835}w^8 - \cdots$$

$$w = \sqrt{-3}\,(z-1)^{\frac{1}{2}} - \frac{3\sqrt{-3}}{5}(z-1)^{3/2} + \cdots$$

For the function involving $\coth w$, $B_0 = \infty$, $W_0 = 0$, and

$$z = w^{-2} + \tfrac{1}{3} - \tfrac{1}{15}w^2 + \tfrac{2}{945}w^4 - \tfrac{1}{4725}w^6 + \cdots$$

$$w = z^{-1/2}\,(1 + \tfrac{1}{12}z^{-1} + \cdots)$$

2. Computation of $w(z)$. We have

$$\tanh(w + k) = wz$$

$$w' \operatorname{sech}^2(w + k) = w'z + w$$

and consequently

$$\frac{dw}{dz} = \frac{w}{1 - z - z^2w^2} \tag{1}$$

[4] Since $W_0 = 0$, from here on we take $n > 0$.

It is also of interest to have w in terms of $1/z = t$. The differential equation in this case becomes

$$\frac{dw}{dt} = \frac{w}{w^2 + t - t^2} \tag{2}$$

Expansions for w in powers of z [or t] can be obtained from equation (1) [or (2)] by standard methods. If $w = a_0 + a_1 z + a_2 z^2 + \cdots$ is to be a solution of (1) and $w^2 = A_0 + A_1 z + A_2 z^2 + \cdots$, then it is convenient to compute the a_n by the following recursion formulae:

$$A_n = a_0 a_n + a_1 a_{n-1} + \cdots + a_n a_0$$

$$a_{n+1} = a_n + \frac{1}{n+1}\left[a_1 A_{n-2} + 2a_2 A_{n-3} + \cdots + (n-1)\, a_{n-1} A_0\right]$$

In this way the coefficients a_n of the powers of z in the expansion of w were obtained as polynomials in a where $a_0 = ai$. They are:

$$a_0 = ia, \qquad a_1 = ia, \qquad a_2 = ia,$$

$$a_3 = i(a - \tfrac{1}{3}a^3), \qquad a_4 = i(a - \tfrac{4}{3}a^3),$$

$$a_5 = i(a - \tfrac{10}{3}a^3 + \tfrac{1}{5}a^5), \qquad a_6 = i(a - \tfrac{20}{3}a^3 + \tfrac{23}{15}a^5),$$

$$a_7 = i(a - \tfrac{35}{3}a^3 + \tfrac{23}{15}a^5 - \tfrac{1}{7}a^7),$$

$$a_8 = i(a - \tfrac{56}{3}a^3 + \tfrac{308}{15}a^5 - \tfrac{178}{105}a^7),$$

$$a_9 = i(a - 28a^3 + \tfrac{266}{5}a^5 - \tfrac{3272}{315}a^7 + \tfrac{1}{9}a^9),$$

$$a_{10} = i(a - 40a^3 + \tfrac{902}{5}a^5 - \tfrac{2872}{63}a^7 + \tfrac{502}{315}a^9).$$

Now let $w(t) = b_0 + b_1 t + b_2 t^2 + \cdots$ and $w^2 = B_0 + B_1 t + B_2 t^2 + \cdots$ where $w(t)$ is to satisfy (2).

The recursion formulae become:

$$B_n = b_0 b_n + b_1 b_{n-1} + \cdots + b_n b_0$$

$$b_{n+1} = \frac{1}{b_0^2(n+1)}\left[(n-1)(b_{n-1} - b_n) - b_1 B_n - 2b_2 B_{n-1} - \cdots - n b_n B_1\right].$$

From this one can obtain the following expressions for the b_n's in terms of $b = 1/ib_0$:

$$b_0 = i(-1/b), \qquad b_1 = ib, \qquad b_2 = ib^3,$$

$$b_3 = i(-\tfrac{1}{3}b^3 + 2b^5), \qquad b_4 = i(-\tfrac{4}{3}b^5 + 5b^7),$$

$$b_5 = i(\tfrac{1}{5}b^5 - 5b^7 + 14b^9), \qquad b_6 = i(\tfrac{23}{15}b^7 - \tfrac{56}{3}b^9 + 42b^{11}),$$

$$b_7 = i(-\tfrac{1}{7}b^7 + \tfrac{2744}{315}b^9 - 70b^{11} + 132b^{13}),$$

$$b_8 = i(-\tfrac{178}{105}b^9 + 44\,b^{11} - 264\,b^{13} + 429\,b^{15}),$$

$$b_9 = i(\tfrac{1}{9}b^9 - \tfrac{818}{63}b^{11} + 209\,b^{13} - 1001\,b^{15} + 1430\,b^{17}),$$

$$b_{10} = i(\tfrac{502}{315}b^{11} - \tfrac{15796}{189}b^{13} + \tfrac{43043}{45}b^{15} - \tfrac{11440}{3}b^{17} + 4862\,b^{19}).$$

For the equation $(\tanh w)/w = z$, $a_0 = \pm\pi i, \pm 2\pi i, \cdots$ and $b_0 = \pm\pi i/2, \pm 3\pi i/2, \cdots$, while for the equation $(\coth w)/w = z$ the values of a_0 and b_0 are interchanged. The numerical values of the coefficients for the various branches calculated are listed below.[5]

Besides these expansions, developments of modifications of w were used. The auxiliary functions, depending on the nature of the branch points of $w(z)$, have already been explained in Section I. The coefficients were found as follows. For the inverse of $(\tanh w)/w$, $w^2(t) = B_0 + B_1 t + B_2 t^2 + \cdots$ was obtained by letting $b_0 = -\pi i/2$ and using the above recursion formulae. For the inverse of $(\coth w)/w$, $w(t)$ has a branch point at $t = 0$ for the branches having $w(0) = 0$. Hence $w^2(t)$ which is analytic at $t = 0$, was obtained from the differential equation

$$\frac{d(w^2)}{dt} = \frac{2w^2}{w^2 + t - t^2}.$$

Then the expansion $w(t) = t^{\frac{1}{2}}(1 + d_1 t + d_2 t^2 + \cdots)$ can be obtained by taking the square root of the series for w^2/t.

To check values of $w(z)$ near the circle of convergence of a series, or to obtain more accurate values, Newton's method was used. Thus if w_1 is an approximation to $w(z)$ and $F(w_1) = z_1$, then

$$w_2 = w_1 + (z - z_1)w'(z_1) = w_1 + \frac{(z - z_1)w_1}{1 - z_1 - z_1^2 w_1^2}$$

is the next approximation.[6]

In computing values of $F(w)$, the identities

$$\frac{\tanh(R + iI)}{R + iI} = \frac{(R \sinh 2R + I \sin 2I) + i(R \sin 2I - I \sinh 2R)}{(R^2 + I^2)(\cosh 2R + \cos 2I)}$$

and

$$\frac{\coth(R + iI)}{R + iI} = \frac{(R \sinh 2R - I \sin 2I) - i(R \sin 2I + I \sinh 2R)}{(R^2 + I^2)(\cosh 2R - \cos 2I)}$$

were used.

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[5] The last two figures of each coefficient are not guaranteed. But, the next to the last figure has a very high probability of being correct, while the last figure is probably correct to within several units. The user should also note the change in the meaning of the notation a_n, b_n, when it occurs below to designate the numerical coefficients (done to avoid too much notation).

[6] Near a branch point z_0, it is better to let $z - z_0 = u^2$, $w(z) = v(u)$ and to base the successive approximations upon the differential equation

$$\frac{dv}{du} = \frac{2uv}{1 - v^2 - z_0 - (z_0^2 + 2z_0 u^2 + u^4)v^2}.$$

In fact, this expression for dv/du yields the recursion formula for finding all the coefficients of the Taylor series for v in powers of u; but the method of successive approximation involves less labor than the calculation of those complex coefficients of the powers of u, should one want only a few isolated values of $w(z)$ near z_0.

Coefficients of Expansions of w defined by $(\tanh w)/w = z$

A. Expansions in terms of $1/z$: $w(z) = i \sum_{n=0}^{\infty} a_n (1/z)^n$, $w^2(z) = \sum_{n=0}^{\infty} b_n (1/z)^n$

BRANCH I: $f(w) = w^2(z)$		BRANCH I:		BRANCH II:	
b_0	−2.4674011003	a_0	−1.5707963268	a_0	−4.7123889804
b_1	2.0000000000	a_1	.6366197724	a_1	.2122065908
b_2	.4052847347	a_2	.2580122755	a_2	.009556010202
b_3	.0583216090	a_3	.1231327813	a_3	−.002324691166
b_4	.0043402394	a_4	.07247537321	a_4	−.0004768730149
b_5	−.0003807785	a_5	.04947722001	a_5	.000001390670261
b_6	−.0001563188	a_6	.03673321169	a_6	.00001507449506
b_7	−.0000126336	a_7	.02869155963	a_7	.000002316132504
b_8	.0000032154	a_8	.02321335257	a_8	−.00000016665062423
b_9	.0000010176	a_9	.019285	a_9	−.000000118062
b_{10}	.0000000549	a_{10}	.016350	a_{10}	−.000000014339

BRANCH III:		BRANCH IV:	
a_0	−7.8539816340	a_0	−10.9955742876
a_1	.1273239545	a_1	.09094568177
a_2	.002064098204	a_2	.0007522223775
a_3	−.0006211089352	a_3	−.0002382973538
a_4	−.00004190354685	a_4	−.000008038322914
a_5	.000004103177819	a_5	.0000009929999299
a_6	.0000006736092579	a_6	.00000007110885320
a_7	−.00000001056295589	a_7	−.000000003886334515
a_8	−.000000009061813640	a_8	−.000000000561300971
a_9	−.00000000042756	a_9	.0000000000074297
a_{10}	.000000000095214	a_{10}	.00000000000408136

B. Expansions in terms of z. $w(z) = i \sum_{n=0}^{\infty} a_n z^n$

BRANCH I:		BRANCH II:		BRANCH III:		BRANCH IV:	
a_0	−3.141592654	a_0	−6.283185307	a_0	−9.424777961	a_0	−12.56637062
a_1	−3.141592654	a_1	−6.283185307	a_1	−9.424777961	a_1	−12.56637062
a_2	−3.141592654	a_2	−6.283185307	a_2	−9.424777961	a_2	−12.56637062
a_3	7.193833	a_3	76.40022	a_3	269.63	a_3	648.9009
a_4	38.20011	a_4	324.4504	a_4	1106.8	a_4	2633.303
a_5	39.00873	a_5	−1137.975	a_5	−12091.	a_5	−56070.73
a_6	−265.6633	a_6	−13368.	a_6	-10845×10^1	a_6	−467274.9
a_7	−1209.260	a_7	−5862.	a_7	4675×10^2	a_7	5045038.
a_8	−645.375	a_8	45156×10^1	a_8	9561×10^2	a_8	7654804×10^1
a_9	12645.36	a_9	18059×10^3	a_9	-512×10^5	a_9	-370858×10^3
a_{10}	48801.5	a_{10}	-1082×10^4	a_{10}	-756×10^6	a_{10}	-1174825×10^4
a_{11}	−5278.4						
a_{12}	-6699×10^2						
a_{13}	-2171×10^3						
a_{14}	198×10^4						
a_{15}	374×10^5						

SINGULARITIES

For $z = 1$, $w = 0$. If $z = x + iy = r\exp(i\phi)$, $w = R + iI$

x	y	r	ϕ	R	I
.10876	.21344	.23955	63.00°	1.38434	−3.74884
.04128	.13126	.13760	72.54°	1.67610	−6.94998
.02191	.09382	.09635	76.85°	1.85838	−10.11926
.01368	.07280	.07407	79.36°	1.99157	−13.27727

As $n \to \infty$, $R_n \to \frac{1}{2}\log[(4n+1)\pi]$, $I_n \to -(n+\frac{1}{4})\pi$, $x_n \to \dfrac{R_n}{R_n^2 + I_n^2}$,

$$y_n \to \frac{-I_n}{R_n^2 + I_n^2},\ r_n \to y_n, \text{ and } \phi_n \to 90°.$$

Coefficients of Expansions of w defined by $(\operatorname{Coth} w)/w = z$

A. Expansions in terms of $1/z$: $w(z) = i\sum_{n=0}^{\infty} a_n(1/z)^n$, $w^2(z) = \sum_{n=0}^{\infty} b_n(1/z)^n$

BRANCH I: $f(w) = w^2(z)$		BRANCH II:		BRANCH III:		BRANCH IV:	
b_0	0	a_0	−3.141592654	a_0	−6.283185307	a_0	−9.424777961
b_1	1.0000000000	a_1	.318309886	a_1	.159154943	a_1	.106103295
b_2	.3333333333	a_2	.0322515344	a_2	.00403144180	a_2	.00119450127
b_3	.0888888889	a_3	−.0042149842	a_3	−.00113957871	a_3	−.00037127192
b_4	.0169312169	a_4	−.0027015498	a_4	−.00012322347	a_4	−.000017173157
b_5	.0011287478	a_5	−.0005322604	a_5	.000008407471	a_5	.000001956420
b_6	−.0006840896	a_6	.0000242281	a_6	.000002812873	a_6	.0000002011252
b_7	−.0003253560	a_7	.000052460	a_7	.000000090610	a_7	−.0000000080952
b_8	−.0000617435	a_8	.000017376	a_8	−.000000047442	a_8	−.0000000020686
b_9	.00000650138	a_9	.000001699	a_9	−.000000006510	a_9	−.00000000001701
b_{10}	.00000863867	a_{10}	−.00000107	a_{10}	.000000000372	a_{10}	.0000000000185
b_{11}	.00000281147						
b_{12}	.00000023227						
b_{13}	−.00000021113						
b_{14}	−.00000011458						
b_{15}	−.00000002443						

B. Expansions in terms of z. $w(z) = i\sum_{n=0}^{\infty} a_n z^n$

BRANCH I:		BRANCH II:		BRANCH III:		BRANCH IV:	
a_0	−1.5707963268	a_0	−4.7123889804	a_0	−7.8539816340	a_0	−10.9955742876
a_1	−1.5707963268	a_1	−4.7123889804	a_1	−7.8539816340	a_1	−10.9955742876
a_2	−1.5707963268	a_2	−4.7123889804	a_2	−7.8539816340	a_2	−10.9955742876
a_3	−.2788681318	a_3	30.16967228	a_3	153.6370427	a_3	432.1357906
a_4	3.596916453	a_4	134.8158561	a_4	638.1101159	a_4	1761.529910
a_5	9.435862593	a_5	−120.6591726	a_5	−4369.890707	a_5	−27725.13714
a_6	9.60432434	a_6	−2870.287869	a_6	−42601.29358	a_6	−237596.8586
a_7	−15.46146550	a_7	−6594.16452	a_7	73746.0678	a_7	1741464.339
a_8	−86.03398955	a_8	40731.60438	a_8	2485360.151	a_8	29296864.77
a_9	−163.1763500	a_9	288000.0853	a_9	4937320.61	a_9	−67710777.3
a_{10}	−26.3169487	a_{10}	28723.394	a_{10}	−122780135.2	a_{10}	−3332557625.

SINGULARITIES

For $z = \infty$, $w = 0$. If $z = x + iy = re^{i\phi}$, $w = R + iI$

x	y	r	ϕ	R	I
.23688	.29563	.37883	51.30°	1.12536	−2.10620
.06294	.16317	.17489	68.91°	1.55157	−5.35627
.02928	.10951	.11335	75.03°	1.77554	−8.53668
.01706	.08200	.08376	78.25°	1.92940	−11.69918

As $n \to \infty$, $R_n \to \frac{1}{2}\log[(4n-1)\pi]$, $I_n \to -(n-\frac{1}{4})\pi$, $x_n \to \dfrac{R_n}{R_n^2 + I_n^2}$, $y_n \to \dfrac{-I_n}{R_n^2 + I_n^2}$, $r_n \to y_n$, and $\phi_n \to 90°$.

REPEATED INTEGRALS OF BESSEL FUNCTIONS AND THE THEORY OF TRANSIENTS IN FILTER CIRCUITS

By J. C. Jaeger

1. Introduction. It was remarked recently (Jaeger [6]) that simple, explicit solutions of certain problems in the theory of filter circuits could be given in terms of the repeated integrals of Bessel functions.

In this paper the application of these functions to the theory of some non-dissipative high-pass and band-pass filters is discussed. The solution for the simple high-pass filter was found long ago by Carson and Zobel [3] to involve a sum of repeated integrals of Bessel functions of order zero: this solution is here regarded as a new mathematical function, the "high-pass filter function", and is tabulated. It is then shown that these functions are of equal importance to the Bessel functions in the theory of filter circuits, in the sense that the solutions for certain more complicated ladder networks can be expressed as integrals involving high-pass filter functions in the same way that the solutions for other related ladder networks involve Bessel functions.

In this way it is found that the solutions of a number of transient problems in the ten simplest semi-infinite ladder networks can be expressed as single integrals involving tabulated functions.

2. The repeated integrals of Bessel Functions. We write as in [6]

$$Ji_{n,r}(t) = \int_0^t dt \int_0^t dt \cdots \int_0^t J_n(t)\, dt, \qquad (1)$$

for the r-ple repeated integral of $J_n(t)$. Here only the case $n = 0$ will ordinarily be required: values of $Ji_{0,r}(t)$ are given in Table I, these having been calculated from the Tables in Gray and Mathews [5] with the corrections noted in Bateman and Archibald [1].

3. Laplace Transforms. We write

$$L\{x(t)\} = \int_0^\infty e^{-st} x(t)\, dt, \qquad (2)$$

for the Laplace transform of $x(t)$, where s is a number whose real part is large enough to make the integral convergent. The following Laplace Transforms will be required:

$$L\left\{\frac{t^n}{n!}\right\} = \frac{1}{s^{n+1}}, \qquad n = 0, 1, 2, \cdots \qquad (3)$$

$$L\{J_n(t)\} = \frac{\{\sqrt{(s^2+1)} - s\}^n}{\sqrt{(s^2+1)}}, \qquad n > -1 \qquad (4)$$

$$L\{Ji_{n,r}(t)\} = \frac{\{\sqrt{(s^2+1)} - s\}^n}{s^r\sqrt{(s^2+1)}}, \qquad n > -1 \qquad (5)$$

TABLE I

t	$2^{-1}Ji_{0,1}(t)$	$2^{-1}Ji_{0,2}(t)$	$2^{-1}Ji_{0,3}(t)$	$2^{-1}Ji_{0,4}(t)$	$2^{-1}Ji_{0,5}(t)$	$2^{-1}Ji_{0,6}(t)$	$2^{-1}Ji_{0,7}(t)$
0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
1	0.45986520	0.11991996	0.02032169	0.00256134	0.00025735	0.00002151	0.00000154
2	0.71288515	0.42452274	0.15113708	0.03903233	0.00795255	0.00134090	0.00019305
3	0.69378363	0.78638122	0.45430322	0.18281103	0.05709552	0.01463919	0.00319059
4	0.51236708	1.09077749	0.92744415	0.52189437	0.22339859	0.07772498	0.02285815
5	0.35765596	1.30361382	1.52931129	1.13166876	0.62423677	0.27694854	0.10322806
6	0.35311061	1.47435762	2.22388958	2.06644096	1.40929348	0.76587133	0.34763798
7	0.47732016	1.67881549	3.00954677	3.37055370	2.75218536	1.77908970	0.95596945
8	0.60537342	1.95222098	3.91459568	5.09588188	4.84995043	3.64370311	2.26804271
9	0.62613322	2.26564798	4.96862865	7.31013494	7.92949743	6.79246215	4.81301828
10	0.53350565	2.55884639	6.17671793	10.09034950	12.25444176	11.78055134	9.36627357
11	0.41989944	2.79560648	7.51773706	13.50903744	18.12634050	19.30454476	17.01533284
12	0.38706109	2.99270788	8.96550799	17.62576189	25.87987554	30.22035067	29.23515040
13	0.45721124	3.20040673	10.51229791	22.49090044	35.87682259	45.55818609	47.97164910
14	0.56033892	3.45555938	12.17410474	28.15718220	48.50423144	66.53542423	75.73212486
15	0.60258097	3.75021712	13.97465612	34.68821723	64.17807355	94.56996292	115.68208955
16	0.55043260	4.04187211	15.92378529	42.15673716	83.34870064	131.29608771	171.74938336
17	0.45627614	4.29343825	18.00960791	50.63484498	106.50313316	178.58244075	248.73681258
18	0.40665286	4.50585275	20.21048051	60.18545571	134.16235242	238.54985944	352.44384496
19	0.44346444	4.71499394	22.51491060	70.86246059	166.87632582	313.58710947	489.79674609
20	0.52918941	4.95772849	24.93127459	82.71895171	205.22134560	406.36453815	668.98601313
21	0.58450469	5.23891777	27.47926523	95.81571695	249.80193454	519.84753338	899.61048989
22	0.56119556	5.52867336	30.17165824	110.22239726	301.25536949	657.31174545	1192.82859677
23	0.48378831	5.79080164	33.00316752	126.01063900	360.25463189	822.36034687	1561.51771438
24	0.42427819	6.01556665	35.95601973	143.24576266	427.50720926	1018.94176154	2020.44240247

We shall also need the following Theorems on the Laplace transformation: if $L\{x(t)\} = x^*(s)$, then, if ω is any real positive number,

(i) $$L\{x(\omega t)\} = \frac{1}{\omega} x^*\left(\frac{s}{\omega}\right); \quad \ldots\ldots(6)$$

(ii) $$L\left\{\int_0^\infty J_0[2(ut)^{\frac{1}{2}}]x(u)\,du\right\} = \frac{1}{s} x^*\left(\frac{1}{s}\right); \quad \ldots\ldots(7)$$

(iii) $$L\left\{\int_0^t J_0[\omega\sqrt{(t^2 - u^2)}]x(u)\,du\right\} = \frac{x^*[\sqrt{(s^2 + \omega^2)}]}{\sqrt{(s^2 + \omega^2)}}; \quad \ldots\ldots(8)$$

(iv) $$L\left\{\int_0^t J_0[2\sqrt{(\omega u(t - u))}]x(u)\,du\right\} = \frac{1}{s} x^*\left(s + \frac{\omega}{s}\right). \quad \ldots\ldots(9)$$

For the results (7), (8), (9), see, e.g., van der Pol [8], McLachlan [7]. There are restrictions on the functions $x(u)$ in these, but they are satisfied in the cases which occur here.

4. The high-pass filter function. We define the "high-pass filter function" $\Phi_n(t)$ by the relation

$$L\{\Phi_n(t)\} = \frac{[\sqrt{(s^2 + 1)} - 1]^n}{s^n\sqrt{(s^2 + 1)}}, \quad \ldots\ldots(10)$$

n being zero or a positive integer. This function may be expressed as a finite series involving the functions $Ji_{0,r}(t)$ by using the relations (Chrystal [4], p. 204)

$$[\sqrt{(s^2+1)}-1]^{2r} = \sum_{m=0}^{r} \frac{r(r+m-1)!2^{2m}s^{2r-2m}}{(r-m)!(2m)!}$$
$$- \sqrt{(s^2+1)} \sum_{m=1}^{r} \frac{(r+m-1)!2^{2m-1}s^{2r-2m}}{(r-m)!(2m-1)!}, \quad \ldots\ldots(11)$$

$$[\sqrt{(s^2+1)}-1]^{2r+1} = -(2r+1) \sum_{m=0}^{r} \frac{(r+m)!2^{2m}s^{2r-2m}}{(2m+1)!(r-m)!}$$
$$+ \sqrt{(s^2+1)} \sum_{m=0}^{r} \frac{(r+m)!2^{2m}s^{2r-2m}}{(2m)!(r-m)!}. \quad \ldots\ldots(12)$$

From (10), (11), (3), and (4) it follows that when n is even, say $n = 2r$,

$$\Phi_{2r}(t) = \sum_{m=0}^{r} \frac{r(r+m-1)!2^{2m}}{(r-m)!(2m)!} Ji_{0,2m}(t) - \sum_{m=1}^{r} \frac{(r+m-1)!2^{2m-1}t^{2m-1}}{(r-m)![(2m-1)!]^2}. \quad \ldots\ldots(13)$$

If n is odd there is a similar result derived from (12).

Values of $\Phi_n(t)$ are given in Table II. The derivative $\Phi_n'(t)$ of $\Phi_n(t)$ is also often needed and is given in Table III. Since $\Phi_n(0) = 1$, it follows that

$$L\{\Phi_n'(t)\} = sL\{\Phi_n(t)\} - 1 = \frac{[\sqrt{(s^2+1)}-1]^n}{s^{n-1}\sqrt{(s^2+1)}} - 1. \quad \ldots\ldots(14)$$

This is equivalent to the statement that

$$L\{\Phi_n'(t) + \delta(t)\} = \frac{[\sqrt{(s^2+1)}-1]^n}{s^{n-1}\sqrt{(s^2+1)}}, \quad \ldots\ldots(15)$$

where $\delta(t)$ is the Dirac delta function.

One relation between $J_n(t)$ and $\Phi_n(t)$ follows immediately from (10), (4), and (7), namely,

$$\Phi_n(t) = \int_0^\infty J_0[2\sqrt{(ut)}]J_n(u)\,du, \quad \ldots\ldots(16)$$

and

$$J_n(t) = \int_0^\infty J_0[2\sqrt{(ut)}]\Phi_n(u)\,du, \quad \ldots\ldots(17)$$

that is, $\Phi_n(\frac{1}{2}t^2)$ and $J_n(\frac{1}{2}u^2)$ are a pair of Hankel transforms, cf. van der Pol [8].

5. The semi-infinite ladder network. We consider always an initially passive, semi-infinite ladder network with mid-series termination, and suppose voltage $V(t)$ is applied to it at $t = 0$. Travelling wave solutions for the finite ladder network with simple terminations can be obtained immediately from the solutions for the semi-infinite network given here.

TABLE II

t	$\Phi_0(t)$	$\Phi_1(t)$	$\Phi_2(t)$	$\Phi_3(t)$	$\Phi_4(t)$	$\Phi_5(t)$	$\Phi_6(t)$	$\Phi_7(t)$
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1	0.76520	0.08027	−0.27544	−0.40949	−0.40285	−0.31522	−0.18941	−0.05496
2	0.22389	−0.42577	−0.37993	−0.11370	0.13809	0.28418	0.31252	0.24902
3	−0.26005	−0.38757	0.03100	0.29960	0.30396	0.14074	−0.06067	−0.20845
4	−0.39715	−0.02473	0.32907	0.24758	−0.02312	−0.22815	−0.26554	−0.16041
5	−0.17760	0.28469	0.25131	−0.08390	−0.27502	−0.20892	−0.00518	0.17472
6	0.15065	0.29378	−0.05449	−0.28313	−0.16547	0.08830	0.23554	0.19578
7	0.30008	0.04536	−0.26940	−0.16942	0.11972	0.24708	0.13573	−0.07130
8	0.17165	−0.21075	−0.21058	0.10070	0.24894	0.10300	−0.12719	−0.22012
9	−0.09033	−0.25227	0.03485	0.24708	0.10767	−0.14460	−0.21756	−0.07606
10	−0.24594	−0.06701	0.22484	0.14399	−0.13145	−0.21744	−0.05143	0.15118
11	−0.17119	0.16020	0.19366	−0.08698	−0.22166	−0.05660	0.16089	0.18647
12	0.04769	0.22588	−0.01065	−0.21862	−0.08814	0.15183	0.18596	0.00751
13	0.20693	0.08558	−0.18982	−0.13680	0.12186	0.19373	0.01287	−0.16976
14	0.17107	−0.12068	−0.18445	0.06661	0.20163	0.04002	−0.16390	−0.15378
15	−0.01422	−0.20516	−0.01249	0.19552	0.08453	−0.14445	−0.16369	0.02476
16	−0.17490	−0.10087	0.16008	0.13628	−0.10597	−0.17803	0.00166	0.16880
17	−0.16985	0.08745	0.17765	−0.04511	−0.18634	−0.03754	0.15729	0.13213
18	−0.01336	0.18669	0.03347	−0.17529	−0.08774	0.13215	0.15016	−0.03740
19	0.14663	0.11307	−0.13342	−0.13793	0.08806	0.16751	−0.00389	−0.16195
20	0.16702	−0.05838	−0.17115	0.02408	0.17349	0.04189	−0.14748	−0.11961
21	0.03658	−0.16901	−0.05208	0.15648	0.09372	−0.11797	−0.14326	0.03953
22	−0.12065	−0.12239	0.10874	0.13976	−0.06958	−0.15979	−0.00009	0.15364
23	−0.16241	0.03242	0.16400	−0.00409	−0.16164	−0.04955	0.13645	0.11325
24	−0.05623	0.15144	0.06830	−0.13830	−0.10048	0.10292	0.13792	−0.03581

TABLE III

t	$\Phi'_0(t)$	$\Phi'_1(t)$	$\Phi'_2(t)$	$\Phi'_3(t)$	$\Phi'_4(t)$	$\Phi'_5(t)$	$\Phi'_6(t)$	$\Phi'_7(t)$
0	0.00000	−1.00000	−2.00000	−3.00000	−4.00000	−5.00000	−6.00000	−7.00000
1	−0.44005	−0.76520	−0.60059	−0.21431	0.21838	0.59138	0.84882	0.97019
2	−0.57672	−0.22389	0.27482	0.53596	0.50221	0.25978	−0.06614	−0.36525
3	−0.33906	0.26005	0.43608	0.19806	−0.16311	−0.40986	−0.44460	−0.28852
4	0.06604	0.39715	0.11551	−0.26099	−0.37966	−0.21474	0.07664	0.31635
5	0.32758	0.17760	−0.24180	−0.32503	−0.07400	0.22501	0.34385	0.23536
6	0.27668	−0.15065	−0.31087	−0.04166	0.25539	0.28928	0.07879	−0.18180
7	0.00468	−0.30008	−0.08604	0.23871	0.25280	−0.00072	−0.24136	−0.27218
8	−0.23464	−0.17165	0.18686	0.24951	−0.01454	−0.24836	−0.22054	0.00602
9	−0.24531	0.09033	0.25922	0.02063	−0.23495	−0.19472	0.05426	0.24040
10	−0.04347	0.24594	0.09055	−0.20374	−0.19743	0.05916	0.23745	0.16201
11	0.17679	0.17119	−0.14362	−0.21613	0.03035	0.22718	0.14354	−0.09460
12	0.22345	−0.04769	−0.22831	−0.02639	0.20894	0.14988	−0.09473	−0.22204
13	0.07032	−0.20693	−0.10084	0.17271	0.17276	−0.07101	−0.21469	−0.09675
14	−0.13338	−0.17107	0.10798	0.19783	−0.02525	−0.20543	−0.10530	0.12237
15	−0.20510	0.01422	0.20522	0.03920	−0.18582	−0.12986	0.10308	0.19754
16	−0.09040	0.17490	0.11133	−0.14526	−0.16122	0.06668	0.19485	0.06337
17	0.09767	0.16985	−0.07723	−0.18545	0.01299	0.18723	0.08807	−0.12735
18	0.18799	0.01336	−0.18539	−0.05358	0.16519	0.12190	−0.09911	−0.17842
19	0.10570	−0.14663	−0.12044	0.12021	0.15541	−0.05591	−0.17961	−0.04813
20	−0.06683	−0.16702	0.04992	0.17527	0.00177	−0.17171	−0.08201	0.12326
21	−0.17112	−0.03658	0.16690	0.06758	−0.14607	−0.11985	0.08986	0.16486
22	−0.11718	0.12065	0.12760	−0.09682	−0.15192	0.04236	0.16765	0.04312
23	0.03952	0.16241	−0.02533	−0.16559	−0.01715	0.15768	0.08195	−0.11521
24	0.15404	0.05623	−0.14885	−0.08038	0.12775	0.12058	−0.07809	−0.15526

Let z' and z be the generalized impedances of the series and shunt elements, respectively, let I_0 be the input current, I_n the current in the n-th series element z', and V_n the voltage drop across the n-th shunt element z, then (cf., e.g., Carslaw and Jaeger [2])

$$L\{I_n\} = L\{V\}\left[zz'\left(1+\frac{z'}{4z}\right)\right]^{-\frac{1}{2}}\left\{\left(1+\frac{z'}{4z}\right)^{\frac{1}{2}}-\left(\frac{z'}{4z}\right)^{\frac{1}{2}}\right\}^{2n}, \quad \ldots\ldots(18)$$

$$L\{V_n\} = L\{V\}\left(1+\frac{z'}{4z}\right)^{-\frac{1}{2}}\left\{\left(1+\frac{z'}{4z}\right)^{\frac{1}{2}}-\left(\frac{z'}{4z}\right)^{\frac{1}{2}}\right\}^{2n-1}. \quad \ldots\ldots(19)$$

We proceed to determine I_n and V_n for a number of simple networks of ladder type.

6. The high-pass filter with series elements capacitance C and shunt elements inductance L. Here

$$z' = 1/Cs; \qquad z = Ls; \qquad z'/4z = \omega^2/s^2, \quad \ldots\ldots(20)$$

where $\omega^2 = 1/4LC$.

For constant applied voltage, $V = E$, $L\{V\} = E/s$, (18) and (19) become

$$L\{I_n\} = E\sqrt{(C/L)}\,\frac{\{\sqrt{(s^2+\omega^2)}-\omega\}^{2n}}{s^{2n}\sqrt{(s^2+\omega^2)}} \quad \ldots\ldots(21)$$

$$L\{V_n\} = E\,\frac{\{\sqrt{(s^2+\omega^2)}-\omega\}^{2n-1}}{s^{2n-1}\sqrt{(s^2+\omega^2)}}. \quad \ldots\ldots(22)$$

It follows from (6) and (10) that, cf. Carson and Zobel [3],

$$I_n = E\sqrt{(C/L)}\,\Phi_{2n}(\omega t) \quad \ldots\ldots(23)$$

$$V_n = E\Phi_{2n-1}(\omega t). \quad \ldots\ldots(24)$$

For an impulsive applied voltage $V = E\delta(t)$, $L\{V\} = E$, it follows in the same way from (15) that

$$I_n = E\omega\sqrt{(C/L)}\,\{\delta(\omega t) + \Phi'_{2n}(\omega t)\} \quad \ldots\ldots(25)$$

$$V_n = E\omega\{\delta(\omega t) + \Phi'_{2n-1}(\omega t)\}. \quad \ldots\ldots(26)$$

Numerical values for the solutions (23)–(26) can be written down from Tables II and III.

It is of some interest to consider cases in which simple expressions are available for the solution when constant voltage E, instead of being applied at midseries termination, is applied to the network through a series impedance z_1.

If $z_1 = 1/2Cs$, that is, the voltage E is applied to a network built up of Π instead of T sections,

$$I_n = \tfrac{1}{2}E\sqrt{(C/L)}\,\{\Phi_{2n}(\omega t) + \Phi_{2n+1}(\omega t)\}. \quad \ldots\ldots(27)$$

If z_1 is the resistance $\sqrt{(L/C)}$

$$I_n = E\sqrt{(C/L)}\,\psi_n(\omega t) \quad \ldots\ldots(28)$$

where $\psi_n(t)$ can be expressed in terms of the $Ji_{0,n}(t)$ and is given in Table IV.

TABLE IV

t	$\Psi_0(t)$	$\Psi_1(t)$	$\Psi_2(t)$
0	0.5	0.5	0.5
1	0.44005	−0.11083	−0.19182
2	0.28836	−0.16958	0.05353
3	0.11302	−0.01902	0.13079
4	−0.01651	0.09696	0.00260
5	−0.06552	0.09398	−0.09809
6	−0.04611	0.01585	−0.06730
7	−0.00067	−0.05134	0.02711
8	0.02933	−0.05738	0.07543
9	0.02726	−0.01432	0.04152
10	0.00435	0.03032	−0.02537
11	−0.01607	0.03957	−0.05675
12	−0.01862	0.01356	−0.03051
13	−0.00541	−0.01882	0.01936
14	0.00953	−0.02925	0.04391
15	0.01367	−0.01292	0.02510
16	0.00565	0.01176	−0.01365
17	−0.00575	0.02249	−0.03483
18	−0.01044	0.01227	−0.02198
19	−0.00556	−0.00709	0.00894
20	0.00334	−0.01777	0.02810
21	0.00815	−0.01157	0.01989
22	0.00533	0.00384	−0.00516
23	−0.00172	0.01406	−0.02287
24	−0.00642	0.01084	−0.01828

7. The band-pass filter in which the series elements are inductance L_0 and capacitance C_0 in series, while the shunt elements are capacitance C. In this case

$$z' = L_0 s + 1/C_0 s = L_0(s^2 + \omega_0^2)/s,$$

where $\omega_0^2 = 1/L_0C_0$. Also $z = 1/Cs$, and therefore

$$z'/4z = (s^2 + \omega_0^2)/\omega_1^2$$

where $\omega_1^2 = 4/L_0C$.

For constant applied voltage, E, (18) gives

$$L\{I_n\} = E\sqrt{(C/L_0)}\,\frac{\{(s^2 + \omega_0^2 + \omega_1^2)^{\frac{1}{2}} - (s^2 + \omega_0^2)^{\frac{1}{2}}\}^{2n}}{\omega_1^{2n-1}(s^2 + \omega_0^2)^{\frac{1}{2}}(s^2 + \omega_0^2 + \omega_1^2)^{\frac{1}{2}}} \quad \ldots\ldots(29)$$

Therefore, using (4) and (8),

$$I_n = E\omega_1\sqrt{(C/L_0)}\int_0^t J_0[\omega_0\sqrt{(t^2 - u^2)}]J_{2n}(\omega_1 u)\,du. \quad \ldots\ldots(30)$$

For an impulsive applied voltage $E\delta(t)$, we find in the same way

$$I_n = E\omega_1\sqrt{(C/L_0)}\left\{J_{2n}(\omega_1 t) - \omega_0 t\int_0^t \frac{J_1[\omega_0\sqrt{(t^2 - u^2)}]}{\sqrt{(t^2 - u^2)}}J_{2n}(\omega_1 u)\,du\right\}. \quad (31)$$

If the applied voltage is $E \sin \omega_0 t$, that is, a sine wave of one of the cut-off frequencies,

$$I_n = E \frac{\omega_0}{\omega_1} \sqrt{(C/L_0)} \left\{ J i_{2n,2}(\omega_1 t) - \omega_0 t \cdot \int_0^t \frac{J_1[\omega_0 \sqrt{(t^2 - u^2)}]}{\sqrt{(t^2 - u^2)}} J i_{2n,2}(\omega_1 u)\, du \right\}. \quad \ldots\ldots(32)$$

8. The high-pass filter in which the series elements are capacitance C, and the shunt elements are inductance L_0 and capacitance C_0 in series. This is the "inverse" of the filter of §7 in the sense that z' and z are interchanged and so z'/z is inverted.

Writing $\omega_0^2 = 1/L_0C_0$ and $\omega_2^2 = 1/4L_0C$, we find for constant applied voltage E

$$L\{I_n\} = E \sqrt{(C/L_0)} \frac{\{(s^2 + \omega_0^2 + \omega_2^2)^{\frac{1}{2}} - \omega_2\}^{2n}}{(s^2 + \omega_0^2)^n (s^2 + \omega_0^2 + \omega_2^2)^{\frac{1}{2}}}. \quad \ldots\ldots\ldots(33)$$

Therefore, using (15) and (8),

$$I_n = E\omega_2 \sqrt{(C/L_0)} \int_0^t J_0[\omega_0 \sqrt{(t^2 - u^2)}] \{\Phi'_{2n}(\omega_2 u) + \delta(\omega_2 u)\}\, du$$

$$= E \sqrt{(C/L_0)} \left\{ J_0(\omega_0 t) + \omega_2 \int_0^t J_0[\omega_0 \sqrt{(t^2 - u^2)}]\, \Phi'_{2n}(\omega_2 u)\, du \right\}. \quad (34)$$

9. The band-pass filter in which the series elements are inductance L, and the shunt elements are inductance L_0 and capacitance C_0 in parallel. This is the dual of the network of §7, except that it is differently terminated. Writing $\omega_0^2 = 1/L_0C_0$, and $\omega_3^2 = 4/LC_0$, we have

$$z'/4z = (s^2 + \omega_0^2)/\omega_3^2$$

The simplest transient to study in this case is that caused by applied voltage $E \cos \omega_0 t$; ω_0 is one of the cut-off frequencies. For this case we have

$$L\{I_n\} = E \sqrt{(C_0/L)} \frac{\{(s^2 + \omega_0^2 + \omega_3^2)^{\frac{1}{2}} - (s^2 + \omega_0^2)^{\frac{1}{2}}\}^{2n}}{\omega_3^{2n-1} (s^2 + \omega_0^2 + \omega_3^2)^{\frac{1}{2}} (s^2 + \omega_0^2)^{\frac{1}{2}}}, \quad \ldots\ldots(35)$$

and then from (4) and (8)

$$I_n = E\omega_3 \sqrt{(C_0/L)} \int_0^t J_0[\omega_0 \sqrt{(t^2 - u^2)}] J_{2n}(\omega_3 u)\, du. \quad \ldots\ldots(36)$$

A result for constant applied voltage may be derived in the same way but is a little more complicated.

10. The high-pass filter in which the series elements are inductance L_0 and capacitance C_0 in parallel, while the shunt elements are inductance L. This is the dual of the filter of §8 and the inverse of that of §9; its cut-off frequency is

$(\omega_0^2 + \omega_4^2)^{\frac{1}{2}}$. Writing $\omega_0^2 = 1/L_0C_0$, $\omega_4^2 = 1/4LC_0$, we find for applied voltage $E \cos \omega t$

$$L\{I_n\} = E\sqrt{(C_0/L)}\,\frac{\{(s^2+\omega_0^2+\omega_4^2)^{\frac{1}{2}} - \omega_4\}^{2n}}{(s^2+\omega_0^2)^n(s^2+\omega_0^2+\omega_4^2)^{\frac{1}{2}}}. \quad \ldots\ldots\ldots(37)$$

Therefore, using (15) and (8),

$$I_n = E\omega_4\sqrt{(C_0/L)}\int_0^t J_0[\omega_0\sqrt{(t^2-u^2)}]\{\Phi'_{2n}(\omega_4 u) + \delta(\omega_4 u)\}\,du$$

$$= E\sqrt{(C_0/L)}\left\{J_0(\omega_0 t) + \omega_4\int_0^t J_0[\omega_0\sqrt{(t^2-u^2)}]\Phi'_{2n}(\omega_4 u)\,du\right\}. \ldots(38)$$

11. The band-pass filter in which the series elements consist of inductance L and capacitance $1/\omega_0^2 L$ in series, while the shunt elements consist of capacitance C and inductance $1/\omega_0^2 C$ in parallel. Writing $\omega_5^2 = 4/LC$ we find for constant applied voltage E,

$$L\{I_n\} = E\sqrt{(C/L)}\,\frac{1}{s\omega_5^{2n-1}}\left\{\left(s+\frac{\omega_0^2}{s}\right)^2 + \omega_5^2\right\}^{-\frac{1}{2}}$$

$$\left[\left\{\left(s+\frac{\omega_0^2}{s}\right)^2 + \omega_5^2\right\}^{\frac{1}{2}} - \left(s+\frac{\omega_0^2}{s}\right)\right]^{2n}. \quad \ldots\ldots(39)$$

Therefore, using (9) and (4),

$$I_n = E\sqrt{C/L}\,\omega_5\int_0^t J_0[2\omega_0\sqrt{\{u(t-u)\}}]J_{2n}(\omega_5 u)\,du. \quad \ldots\ldots(40)$$

12. The band-suppression filter in which the series elements consist of capacitance C and inductance $1/\omega_0^2 C$ in parallel, while the shunt elements consist of inductance L and capacitance $1/\omega_0^2 L$ in series. Writing $\omega_6^2 = 1/4LC$ we find for constant applied voltage E,

$$L\{I_n\} = E\sqrt{(C/L)}\,\frac{1}{s}\left(s+\frac{\omega_0^2}{s}\right)^{1-2n}\left\{\left(s+\frac{\omega_0^2}{s}\right)^2 + \omega_6^2\right\}^{-\frac{1}{2}}$$

$$\left[\left\{\left(s+\frac{\omega_0^2}{s}\right)^2 + \omega_6^2\right\}^{\frac{1}{2}} - \omega_6\right]^{2n} \quad (41)$$

It follows from (15) and (9) that

$$I_n = E\omega_6\sqrt{(C/L)}\int_0^t J_0[2\omega_0\sqrt{\{u(t-u)\}}]\{\Phi'_{2n}(u\omega_6) + \delta(u\omega_6)\}\,du$$

$$= E\sqrt{(C/L)}\left\{1 + \omega_6\int_0^t J_0[2\omega_0\sqrt{\{u(t-u)\}}]\Phi'_{2n}(u\omega_6)\,du\right\}. \quad (42)$$

13. The relation of the solutions of §§7–12 to the general theory of the ladder network must now be considered. The two simplest ladder networks are the

simple low- and high-pass filters, in which there is a single element, an inductance or a capacitance in each branch. The high-pass case has been discussed in §6, the Bessel function solutions for the low-pass filter are too well known to need mention. Next in order of complexity come eight cases in which there is a single element, an inductance or a capacitance, in one set of branches of the network and in the other set a series or parallel combination of an inductance and a capacitance. These eight circuits are, as remarked above, related to each other by duality, inversion, and by the process called reflection by van der Pol [8]. Four of these circuits have been discussed in §§7–10 and in these cases the solutions of many important transient problems can be written down by (8) as single integrals involving Bessel or high-pass filter functions which are easy to evaluate numerically. To write down solutions involving single integrals of this type for the remaining four circuits requires a simple result similar to (7), (8), and (9) for the function $x^*[\sqrt{(1 + a^2/s^2)}]$ and this does not seem to exist. These four circuits are treated briefly in §§14–17 by a different method. The next ladder networks in order of complexity are those in which each branch contains both an inductance and a capacitance, the natural frequencies of the branches being equal. There are two circuits of this type, and simple solutions for these have been found §§11–12.

14. The band-pass filter in which the series elements consist of inductance L_0 and capacitance C_0 in series, while the shunt elements consist of inductance L. Writing $\omega_0^2 = 1/L_0C_0$, $k^2 = 4L/L_0$, we find for constant applied voltage E,

$$L\{I_n\} = E\sqrt{(C_0/L)}\,\frac{\omega_0}{k^{2n-1}}\,\frac{\{[1 + k^2 + \omega_0^2/s^2]^{\frac{1}{2}} - [1 + \omega_0^2/s^2]^{\frac{1}{2}}\}^{2n}}{s^2(1 + \omega_0^2/s^2)^{\frac{1}{2}}(1 + k^2 + \omega_0^2/s^2)^{\frac{1}{2}}} \quad \ldots\ldots(43)$$

Expanding the numerator of (43) by (11) we find

$$L\{I_n\} = E\sqrt{(C_0/L)}\sum_{m=0}^{n}\frac{n\omega_0(n + m - 1)!\,2^{2m}k^{1-2m}}{(2m)!(n - m)!}\,\frac{(1 + \omega_0^2/s^2)^m}{s^2(1 + \omega_0^2/s^2)^{\frac{1}{2}}(1 + k^2 + \omega_0^2/s^2)^{\frac{1}{2}}}$$

$$- E\sqrt{(C_0/L)}\sum_{m=1}^{n}\frac{\omega_0(n + m - 1)!\,2^{2m-1}k^{1-2m}}{(2m - 1)!(n - m)!}\,\frac{1}{s^2}\left(1 + \frac{\omega_0^2}{s^2}\right)^{m-1}.$$

Therefore, using (5) and Duhamel's theorem,

$$I_n = E\sqrt{(C_0/L)}\sum_{m=0}^{n}\sum_{\nu=0}^{m}\frac{n(n + m + 1)!\,m!\,2^{2m}k^{1-2m}\omega_0}{(1 + k^2)^{\frac{1}{2}+\nu}(2m)!(n - m)!\nu!(m - \nu)!}\int_0^t J_0[\omega_0(t - \tau)] \times Ji_{0,2\nu}\left(\frac{\omega_0\tau}{(1 + k^2)^{\frac{1}{2}}}\right)d\tau \quad (44)$$

$$- E\sqrt{(C_0/L)}\sum_{m=1}^{n}\sum_{\nu=0}^{m-1}\frac{(n + m - 1)!(m - 1)!\,2^{2m-1}k^{1-2m}\omega_0^{2\nu+1}t^{2\nu+1}}{(2m - 1)!(n - m)!\nu!(m - \nu - 1)!(2\nu + 1)!}$$

15. The band-pass filter in which the series elements consist of capacitance C, and the shunt elements of inductance L_0 and capacitance C_0 in parallel. Writing $\omega_0^2 = 1/L_0C_0$, $k^2 = 4C/C_0$, we find for applied voltage $E \cos \omega_0 t$

$$L\{I_n\} = E\sqrt{\left(\frac{C}{L_0}\right)}\frac{1}{\omega_0 k^{2n-1}}\frac{\{[1 + k^2 + \omega_0^2/s^2]^{\frac{1}{2}} - [1 + \omega_0^2/s^2]^{\frac{1}{2}}\}^{2n}}{(1 + \omega_0^2/s^2)^{\frac{1}{2}}(1 + k^2 + \omega_0^2/s^2)^{\frac{1}{2}}}, \quad (45)$$

and I_n follows as in §14.

16. The low-pass filter in which the series elements consist of inductance L, and the shunt elements of inductance L_0 and capacitance C_0 in series. Writing $\omega_0^2 = 1/L_0C_0$, $k^2 = 4L_0/L$, we find for constant applied voltage E,

$$L\{I_n\} = E\sqrt{(C_0/L)}\,\frac{\omega_0}{k^{2n-1}}\,\frac{\{[k^2 + s^2/(s^2 + \omega_0^2)]^{\frac{1}{2}} - [s^2/(s^2 + \omega_0^2)]^{\frac{1}{2}}\}^{2n}}{s(s^2 + \omega_0^2)^{\frac{1}{2}}[k^2 + s^2/(s^2 + \omega_0^2)]^{\frac{1}{2}}} \quad \ldots (46)$$

Using (11) this becomes

$$\begin{aligned} L\{I_n\} = {} & E\sqrt{(C_0/L)}\sum_{m=0}^{n}\frac{n\omega_0(n + m - 1)!\,2^{2m}k^{1-2m}}{(2m)!(n - m)!s^2} \\ & \left(\frac{s^2}{s^2 + \omega_0^2}\right)^{m+\frac{1}{2}}\left[k^2 + \frac{s^2}{s^2 + \omega_0^2}\right]^{-\frac{1}{2}} \qquad (47) \\ & - E\sqrt{(C_0/L)}\sum_{m=1}^{n}\frac{\omega_0(n + m - 1)!\,2^{2m-1}k^{1-2m}}{(2m-1)!(n - m)!s^2}\left(\frac{s^2}{s^2 + \omega_0^2}\right)^{m}. \end{aligned}$$

It follows that

$$\begin{aligned} I_n = {} & E\sqrt{(C_0/L)}\sum_{m=0}^{n}\frac{n(n + m - 1)!\,2^{m+1}k^{1-2m}}{(2m)!(n - m)!(m - 1)!(1 + k^2)^{\frac{1}{2}}} \\ & \int_0^t J_0\left\{\frac{\omega_0 k(t - \tau)}{(1 + k^2)^{\frac{1}{2}}}\right\}\frac{d^m}{d\tau^m}\{\tau^{m-1}\sin\omega_0\tau\}\,d\tau \qquad (48) \\ & - E\sqrt{(C_0/L)}\sum_{m=1}^{n}\frac{(n + m - 1)!\,2^m k^{1-2m}}{(2m - 1)!(n - m)!(m - 1)!}\frac{d^{m-1}}{dt^{m-1}}\{t^{m-1}\sin\omega_0 t\}. \end{aligned}$$

The last remaining case in which the series elements consist of inductance L_0 and capacitance C_0 in parallel, while the shunt elements consist of capacitance C, leads to a similar result.

REFERENCES

1. Bateman, H., and Archibald, R. C.: *Math. Tables and Aids to Computation*, **1**: (1944), 205.
2. Carslaw, H. S., and Jaeger, J. C.: *Operational Methods in Applied Mathematics*, (Oxford, Ed. 2, 1948).
3. Carson, J. R., and Zobel, O. J.: *Bell Syst. Tech. Journ.* **2**: (1923), 1.
4. Chrystal, G.: *Textbook of Algebra*, Vol. 2, (Ed. 2, 1906).
5. Gray and Mathews: *Treatise on Bessel Functions*, (Macmillan, Ed. 2, 1922).
6. Jaeger, J. C.: *Quart. App. Math.*, **4**: (1946), 302.
7. McLachlan, N. W.: *Math. Gazette*, **30**: (1946), 85.
8. van der Pol, B.: *Physica*, **1**: (1934), 521.

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ON TWO-DIMENSIONAL NON-STEADY MOTION OF A SLENDER BODY IN A COMPRESSIBLE FLUID

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1. Introduction. Many problems of aerodynamics involve flow around a slender body. It is known that in such cases a perturbation approach often leads to linear differential equations. Recently, von Karman[1] pointed out that for steady motions in the transonic range, certain non-linear terms have to be retained even in a perturbation theory. This fact was also discussed by C. Kaplan[2] together with a comparison with results of his earlier iteration calculation. Tsien[3] has carried the same point of view to hypersonic flow. Quite recently, I. E. Garrick,[4] during his discussion of flutter in the transonic range, wrote down an equation for the non-steady disturbance potential which contains the same non-linear term as the equation derived by von Karman. Garrick, however, gave no discussion of the size of the different terms for different flow conditions and of the appropriate similarity laws. In this article, we shall present a simple unified treatment of the unsteady two-dimensional polytropic potential flow around a slender body. We shall give for different situations a systematic estimate of the size of the various terms in the differential equation for the velocity potential. This then includes all the known special cases and yields others not discussed by previous authors.

The general conclusions are as follows. Let δ be the larger of the two quantities: the thickness ratio of the body and the ratio of the amplitude of lateral motion of the body and its chord. Let M_∞ be the Mach number at infinity, and let $1/k$ be the characteristic period of time for transient motions; e.g., in the case of oscillating wings, k would be the dimensionless ratio of the product of the angular frequency and the chord length to the free stream velocity. These are the three important parameters.

A. If M_∞ is not large, there are the following possibilities:

If $|1 - M_\infty| = 0(\delta^{2/3})$ and $k = 0(\delta^{2/3})$ (or smaller) we are in the transonic range, and the problem is non-linear. In this case, if $k = o(\delta^{2/3})$, the problem is quasi-stationary.[5] Notice that the comparison between k and $|1 - M_\infty|$ does not enter the problem.

If either or both of the above conditions are not satisfied, that is, if $|1 - M_\infty|$ or/and $k \gg \delta^{2/3}$ the general linearized theory holds, even though M_∞ may become unity. In such cases, if $k = o(|1 - M_\infty|)$ the problem is quasi-stationary.

[1] Th. von Karman: *The Similarity Law of Transonic Flow*, J. Math. Phys. **26**, pp. 182–190 (1947).

[2] C. Kaplan: *On Similarity Rules for Transonic Flows*, NACA Tech. Note No. 1527, 16 pp. (1948).

[3] H. S. Tsien: *Similarity Laws of Hypersonic Flows*, J. Math. Phys. **25**, pp. 247–251 (1946)

[4] I. E. Garrick: *A Survey of Flutter*, Papers presented at the NACA-University Conference in Aerodynamics, pp. 291–292 (June 1948).

[5] The notation $y = 0(x)$ and $y = o(x)$ are used in the following conventional sense. If $y = 0(x)$, $\lim_{x\to 0} y/x$ is bounded or zero. If $y = o(x)$, $\lim_{x\to 0} y/x = 0$.

B. If $M_\infty \gg 1$, there are the following possibilities:

If $M_\infty \ll 1/\delta$ and $1/k\delta$, the general linearized theory holds.

If either or both of the above conditions are not satisfied, the equations are non-linear and in the first of the two cases the flow is hypersonic. Moreover, we then have no longer a perturbation theory in the sense that the pressure perturbation is not small.

The method of attack is to introduce a parameter for each type of quantity to characterize its size. In this way, one can distinguish small quantities of various orders. One then neglects those quantities which are definitely of higher orders in *comparison* with other terms in the same equation.

To fix our ideas, we shall consider the periodic oscillations of a two-dimensional thin airfoil at small angles of attack. The scheme is obviously capable of including more general cases.

2. General formulation of the problem. The basic equations for unsteady two-dimensional polytropic potential motions are (in dimensionless form),

$$\phi_{tt} + 2\phi_x\phi_{xt} + 2\phi_y\phi_{yt} + \phi_x^2\phi_{xx} + 2\phi_x\phi_y\phi_{xy} + \phi_y^2\phi_{yy} = c^2(\phi_{xx} + \phi_{yy})$$

$$\phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2) + \frac{c^2}{\gamma - 1} = \frac{1}{2} + \frac{1}{(\gamma - 1)M_1^2} \tag{1}$$

where ϕ is the (dimensionless) velocity potential, c is the (dimensionless) velocity of sound, and M_1 is the Mach number of the reference velocity U_1 which may be taken to be either the free stream velocity U_∞ or the critical velocity c_*. The reference length is the chord length b. The non-dimensional time variable t is constructed by multiplying the actual time by U_1 and dividing by b. On the airfoil

$$y = h(x, t), \tag{2}$$

the boundary condition

$$h_t + \phi_x h_x = \phi_y \tag{3}$$

must be satisfied. If the velocity potential is written as

$$\phi = x + \phi' \tag{4}$$

then, at infinity, we have the boundary conditions

$$1 + \phi_x' = \frac{U_\infty}{U_1} = \left\{1 + \frac{1}{(\gamma - 1)M_1^2}\right\}^{\frac{1}{2}} \Big/ \left\{1 + \frac{1}{(\gamma - 1)M_\infty^2}\right\}^{\frac{1}{2}}, \qquad \phi_y' = 0. \tag{5}$$

If the reference velocity is taken to be the critical velocity c_* then

$$M_1 = 1 \tag{6}$$

If it is taken to be the free stream velocity then

$$M_1 = M_\infty \tag{7}$$

In the latter case,

$$\phi'_x = 0, \quad \phi'_y = 0, \quad \text{at infinity} \tag{8}$$

We shall make the latter choice because it enables one to cover arbitrary free-stream Mach number. The former choice imposes an additional restriction that when the magnitude of ϕ'_x at infinity is small compared with unity it must at the same time be of the order of $|1 - M_\infty|$. It is therefore limited to the transonic case. In fact, this additional restriction may be taken to be the criterion for transonic flow. Besides this additional restriction, and certain unessential changes in formulae, the results are not affected otherwise. Thus, it is very easy to include existing discussions of transonic flow in the present development. This will be carried through in some detail when the transonic case is brought up.

3. Introduction of parameters. Let us now introduce dimensionless quantities of the order of magnitude unity:

$$\xi = x, \quad \eta = \lambda y, \quad \tau = kt,$$
$$g(\xi, \tau) = \delta^{-1} h(x, t), \quad f(\xi, \eta, \tau) = \epsilon^{-1} \phi'(x, y, t) \tag{9}$$

where λ, k, δ, ϵ are parameters, which shall specify the orders of magnitude of the various quantities. In particular

$$k = \omega b / U_1 \tag{10}$$

is the dimensionless angular frequency, and δ is the thickness ratio of the airfoil.

The components of velocity and the velocity of sound are

$$u = 1 + \epsilon f_\xi, \qquad v = \epsilon\lambda f_\eta$$
$$c^2 = \frac{1}{M_1^2} - (\gamma - 1)\epsilon[kf_\tau + f_\xi + \tfrac{1}{2}\epsilon(f_\xi^2 + \lambda^2 f_\eta^2)] \tag{11}$$

and the pressure coefficient is

$$C_p = \frac{p - p_1}{\frac{1}{2}\rho_1 U_1^2} = \frac{2}{\gamma M_1^2}\{[1 - (\gamma - 1)\epsilon M_1^2(kf_\tau + f_\xi + \tfrac{1}{2}\epsilon(f_\xi^2 + \lambda^2 f_\eta^2))]^{\gamma/(\gamma-1)} - 1\} \tag{12}$$

4. Theory of small perturbations. The assumption of *small* disturbances demands that the departure of the velocity components *and* the pressure from the reference conditions is small. Thus, we have the conditions

$$\epsilon \ll 1, \quad \epsilon\lambda \ll 1, \tag{13}$$

from the expressions (11) for the velocity components, and the conditions

$$\epsilon M_1^2 \ll 1, \quad (\epsilon\lambda M_1)^2 \ll 1, \quad \epsilon k M_1^2 \ll 1, \tag{13A}$$

from the expression (12) for the pressure coefficient.

The complete equation (1) is

$$k^2 f_{\tau\tau} + 2(1 + \epsilon f_\xi)kf_{\xi\tau} + 2(\epsilon\lambda^2)kf_\eta f_{\eta\tau} + (1 + 2\epsilon f_\xi + \epsilon^2 f_\xi^2)f_{\xi\xi} + 2(\epsilon\lambda^2)(1 + \epsilon f_\xi)f_{\xi\eta}$$
$$+ (\epsilon\lambda^2)^2 f_\eta^2 f_{\eta\eta} = \left\{\frac{1}{M_1^2} - (\gamma - 1)\epsilon[kf_\tau + f_\xi + \tfrac{1}{2}\epsilon(f_\xi^2 + \lambda^2 f_\eta^2)]\right\}(f_{\xi\xi} + \lambda^2 f_{\eta\eta}). \tag{14}$$

Under the condition (13) of small disturbances, *without any knowledge of the relative sizes of the parameters*, the above equation simplifies to

$$k^2 f_{\tau\tau} + 2kf_{\xi\tau} + \left[1 - \frac{1}{M_1^2} + \epsilon(\gamma + 1)f_\xi\right] f_{\xi\xi} - \frac{\lambda^2}{M_1^2} f_{\eta\eta} = 0. \tag{15}$$

In this equation the orders of magnitude of the different terms are not necessarily the same and this allows further simplification of the equation, to be discussed subsequently. However all terms in (14) that are neglected are definitely small compared with some term retained in (15). For instance, the term $k\epsilon(\gamma - 1)f_\tau f_{\xi\xi}$ is small compared with $2kf_{\xi\tau}$ and is thus neglected.

Boundary conditions. The boundary condition (3) is

$$\delta[kg_\tau + (1 + \epsilon f_\xi)g_\xi] = \epsilon\lambda f_\eta; \quad \eta = 0, 0 < \xi < 1$$

which simplifies to

$$\eta = 0, \quad 0 < \xi < 1; \qquad \epsilon\lambda f_\eta = \delta[kg_\tau + g_\xi] \tag{16}$$

Eq. (16) implies that the larger of the two quantities $k\delta$ and δ must be of the order of $\epsilon\lambda$. Note that the possibility of satisfying the boundary condition at $\eta = 0$ rather than at the actual boundary depends on the fact that $\eta \ll 1$ at the actual boundary in all cases, as can be verified *à posteriori*. Besides this fact we must *assume* that the solution is regular near $\eta = 0$ in such a way that this simplification is permitted.

The boundary conditions at infinity do not impose any restriction when taken in the form (8). That is, when $M_1 = M_\infty$, the conditions at infinity are

$$f_\xi(-\infty, \eta) = f_\eta(-\infty, \eta) = 0 \tag{17}$$

The formula (12) for the pressure coefficient becomes

$$C_p = 2\epsilon[kf_\tau + f_\xi + \tfrac{1}{2}\epsilon\lambda^2 f_\eta^2] \tag{18}$$

Equation (15)–(18) are the basic equations for the small perturbation theory of a two-dimensional potential motion. Additional simplifications may be carried out for special cases. In fact, all the known perturbation theories are obtainable from them by such specialisations, as will be shown in the following sections.

Similarity of disturbance prevails when two cases have identical equations and boundary conditions.

5. Further simplification in special cases. We shall carry out some further reduction in the individual cases to bring out the important parameters involved. We shall take $M_1 = M_\infty$ in our discussions.

Our first step is the elimination of the parameter λ, as follows. We distinguish two cases by considering the boundary condition (16). We set

$$\text{(A)} \quad \lambda = \delta/\epsilon \quad \text{when } k = 0(1) \tag{19}$$

$$\text{(B)} \quad \lambda = \delta k/\epsilon \quad \text{when } 1 \ll k \tag{20}$$

In the first case, (15), (16), and (19) give

$$f_{\eta\eta} = \frac{\epsilon^2 M_\infty^2 k^2}{\delta^2} f_{\tau\tau} + \frac{2k\epsilon^2 M_\infty^2}{\delta^2} f_{\xi\tau} + \frac{\epsilon^2}{\delta^2}(M_\infty^2 - 1) f_{\xi\xi} + (\gamma + 1)\frac{\epsilon^3}{\delta^2} M_\infty^2 f_\xi f_{\xi\xi} \tag{21}$$

$$\eta = 0, \quad 0 < \xi < 1; \qquad f_\eta = g_\xi + k g_\tau \tag{22}$$

and

$$C_p = 2\epsilon\left[k f_\tau + f_\xi + \tfrac{1}{2}\frac{\delta^2}{\epsilon} f_\eta^2\right]. \tag{23}$$

The conditions (13) for small disturbances become

$$\epsilon \ll 1, \quad \delta \ll 1, \quad \epsilon M_\infty^2 \ll 1, \quad M_\infty \delta \ll 1 \tag{24}$$

In the second case, we have

$$f_{\eta\eta} = \frac{\epsilon^2 M_\infty^2}{\delta^2} f_{\tau\tau} - \frac{\epsilon^2}{k^2\delta^2} f_{\xi\xi} \tag{25}$$

$$\eta = 0, \quad 0 < \xi < 1; \quad f_\eta = g_\tau \tag{26}$$

$$C_p = 2\epsilon k f_\tau + k^2\delta^2 f_\eta^2 \tag{27}$$

The conditions (13) become

$$\epsilon \ll 1, \quad k\delta \ll 1, \quad k\delta M_\infty \ll 1, \quad \epsilon k M_\infty^2 \ll 1 \tag{28}$$

To have a meaningful boundary value problem, we must retain the term with $f_{\eta\eta}$ in equations (21) and (25). That is why these equations are solved explicitly for $f_{\eta\eta}$. Further simplification depends on the order of magnitude of the coefficients. These will then appear as certain conditions on the parameters δ, k, and M_∞ of the problem:

Class A, $k = 0(1)$. In this class of problems, there are the four parameters

$$\left.\begin{aligned} c_1 &= (\gamma + 1)\frac{\epsilon^3}{\delta^2} M_\infty^2, & c_2 &= \frac{\epsilon^2}{\delta^2}(M_\infty^2 - 1), \\ c_3 &= \frac{2k\epsilon^2 M_\infty^2}{\delta^2}, & c_4 &= \frac{\epsilon^2 k^2 M_\infty^2}{\delta^2} \end{aligned}\right\} \tag{29}$$

But only the order of magnitude of three of them need be discussed, since $c_4 = 0(c_3)$. The various cases may be distinguished further as follows:

Case 1. $c_1 = 1, \quad c_2 = 0(1), \quad c_3 = 0(1)$ (30)

Case 2. $c_1 \ll 1, \quad |c_2| = 1, \quad c_3 = 0(1)$ (31)

Case 3. $c_1 \ll 1, \quad |c_2| \ll 1, \quad c_3 = 1$ (32)

That is, c_1, c_2 and c_3 are successively taken to be the principal parameter. Notice that in Case 2 we impose the restriction $c_1 \ll 1$, rather than let $c_1 = 0(1)$, because the case $c_1 \sim 1$ is essentially included in Case 1. A similar remark applies to Case 3.

Let us work out Case 1 in some detail and then summarize the results. The other cases can be treated in a very similar manner. We have, from (29) and (30),

$$\left.\begin{aligned} \epsilon &= (\gamma+1)^{-1/3}(\delta/M_\infty)^{2/3} \\ c_2 &= (M_\infty^2-1)[(\gamma+1)M_\infty\delta]^{-2/3} = 0(1) \\ c_3 &= 2kM_\infty^2[(\gamma+1)M_\infty\delta]^{-2/3} = 0(1) \\ c_4 &= k^3M_\infty^2[(\gamma+1)M_\infty\delta]^{-2/3} \end{aligned}\right\} \tag{33}$$

Thus, it is clear that one must have

$$M_\infty^2 - 1 = 0[(\gamma+1)M_\infty\delta]^{2/3}, \qquad k = 0[(\gamma+1)M_\infty\delta]^{2/3}$$

or, since $M_\infty \sim 1$,

$$M_\infty - 1 = 0(\delta^{2/3}), \qquad k = 0(\delta^{2/3}) \tag{34}$$

If one checks back with the condition (24), one sees that they are all satisfied when $\delta \ll 1$. The similarity parameters are essentially c_2 and c_3, because c_4 is negligible in view of (34).

The result may be summarized as follows:

Case 1. *Transonic case.* When $\delta \ll 1$ and the conditions (34) are satisfied, we have the transonic case with always a non-linear equation. The fundamental equations are

$$\left.\begin{aligned} &\frac{2k}{[(\gamma+1)\delta]^{2/3}}f_{\tau\tau} + \left\{\frac{M_\infty-1}{[(\gamma+1)\delta]^{2/3}} + f_\xi\right\}f_{\xi\xi} - f_{\eta\eta} = 0 \\ &\eta = 0, \qquad 0 < \xi < 1; \qquad f_\eta = g_\xi \\ &C_p = 2\delta^{2/3}(\gamma+1)^{-1/3}f_\xi \end{aligned}\right\} \tag{35}$$

In this case, there are the two parameters

$$\frac{k}{[(\gamma+1)\delta]^{2/3}} \quad \text{and} \quad \frac{M_\infty-1}{[(\gamma+1)\delta]^{2/3}}.$$

If either or both of them are large compared with unity, we pass out of the present case. When $k \ll \delta^{2/3}$, the problem is quasi-stationary.

Case 2. *Subsonic or supersonic case.* The condition $|c_2| = 1$ leads to the following relation for the order of magnitude of the velocity disturbance

$$\epsilon = \delta\,|1 - M_\infty^2|^{-1} \tag{37}$$

The conditions (24) and (31) become

$$\delta \ll 1, \qquad \delta M_\infty \ll 1$$

and (38)

$$k = 0(1 - M_\infty^{-2}), \qquad |1 - M_\infty^2| \gg \delta^{2/3}$$

When these are satisfied, we have the usual cases of supersonic or subsonic flow.

The fundamental equations are

$$\left.\begin{aligned} k^2 f_{\tau\tau} + 2k f_{\xi\tau} + \left(1 - \frac{1}{M_\infty^2}\right)\left(f_{\xi\xi} + \frac{1 - M_\infty^2}{|\,1 - M_\infty^2\,|} f_{\eta\eta}\right) = 0 \\ \eta = 0, \quad 0 < \xi < 1; \qquad f_\eta = g_\xi + k g_\tau \\ C_p = 2\delta\,|\,1 - M_\infty^2\,|^{-1/2}(f_\xi + k f_\tau) \end{aligned}\right\} \tag{39}$$

The two parameters occurring in the differential equation are k and $1 - M_\infty^{-2}$. The pressure coefficient is further proportional to δ. When $k = 0$, the problem is stationary; when $k \ll |\,1 - M_\infty^2\,|$ the problem is quasi-stationary.

Case 3. *Highly oscillatory case.* The condition $c_3 = 1$ leads to the relation

$$\epsilon = \delta/\sqrt{2kM_\infty}. \tag{40}$$

The conditions (24) and (32) become

$$\delta \ll 1, \qquad \delta M_\infty \ll 1$$

and

$$k \gg \delta^{2/3}, \qquad k \gg \left|\,1 - \frac{1}{M_\infty^2}\,\right| \tag{41}$$

This is the case where the time rate of change predominates. The fundamental equations are

$$\left.\begin{aligned} k f_{\tau\tau} + 2(f_{\xi\tau} - f_{\eta\eta}) = 0 \\ \eta = 0, \qquad 0 < \xi < 1; \qquad f_\eta = g_\xi + k g_\tau \\ C_p = (2\delta/\sqrt{2kM_\infty})(f_\xi + k f_\tau) \end{aligned}\right\} \tag{42}$$

When $k \ll 1$, the terms in k in (42) can be omitted.

Class B, $k \gg 1$. In this class of problems, we have the two parameters

$$c_4 = \epsilon^2 M_\infty^2/\delta^2, \qquad c_5 = \epsilon^2/k^2\delta^2 \tag{43}$$

in Eq. (24). Consequently, there are two cases:

Case 4. $$c_4 = 1, \quad c_5 = 0(1) \tag{44}$$

Case 5. $$c_4 \ll 1, \quad c_5 = 1. \tag{45}$$

Case 4. Highly oscillatory case. The condition $c_4 = 1$ leads to

$$\epsilon = \delta/M_\infty$$

The other condition in (44) and the condition (28) lead to

$$1/k = 0(M_\infty), \qquad k\delta M_\infty \ll 1 \tag{46}$$

The fundamental equations are

$$\left.\begin{aligned} f_{\tau\tau} - f_{\eta\eta} - (1/k^2 M_\infty^2) f_{\xi\xi} = 0 \\ \eta = 0, 0 < \xi < 1; \qquad f_\eta = g_\tau \\ C_p = 2(\delta k/M_\infty) f_\tau \end{aligned}\right\} \tag{47}$$

There is the only parameter kM_∞.

Case 5. Quasi-stationary nearly imcompressible flow.

The condition $c_0 = 1$ leads to

$$\epsilon = k\delta \tag{48}$$

The other condition in (42) and the condition (27) lead to

$$kM_\infty \ll 1; \qquad k\delta \ll 1 \tag{49}$$

The fundamental equations are

$$\left.\begin{gathered} f_{\xi\xi} + f_{\eta\eta} = 0 \\ \eta = 0, \quad 0 < \xi < 1; \quad f_\eta = g_\tau \\ C_p = 2\delta k^2 f_\tau \end{gathered}\right\} \tag{50}$$

The above results are summarized in Table 1.

For convenience of the discussion, the various items are numbered on the left. Item 1 gives the characterizing conditions in terms of the important parameters M_∞, k and δ. The equations which hold in the various cases are indicated in item 2. Item 3 gives the type of formula for the pressure coefficient. The functions F are not the same. Their arguments give the similarity parameters in the individual cases. In item 4, the physical nature of the various cases is described. Finally, it is indicated that, only in one case, is the theory non-linear; i.e., when

$$|1 - M_\infty| = 0(\delta^{2/3}) \text{ and } k = 0(\delta^{2/3}) \tag{51}$$

But it should be emphasized that in this transonic case, the nonsteady effects appear even at relatively low reduced frequencies. Take for instance the case for which the thickness ratio and the amplitude ratio are of the same order of magnitude and where the value for thickness ratio is 5% or $\delta = 0.05$. Then non-steady effects are already important when $k \approx (0.05)^{2/3}$ according to (51). Consider particularly the case of harmonic oscillation, the period T of the corresponding motion is then

$$T = 2\pi b/kU = 46.5b/U$$

By taking $b = 5$ ft. and $U = 1000$ ft./sec., $T = 0.237$ second. This is a relatively slow oscillation. In subsonic or supersonic flow, the non-steady effects of such slow oscillation are negligible as shown by the conditions of case 2 in Table I. But for transonic flow, the non-steady effects are not negligible. This fact must be kept in mind when the stability of aircraft in transonic flight is considered, as it is likely that a quasi-stationary calculation will not be sufficiently accurate.

Linearization. If either of the conditions (51) is not satisfied, the equation is linear. Note also that the boundary conditions and the formula for the pressure coefficient are always linear. Thus, if one includes all the linear terms in the equation, one does not have to distinguish between the cases 2 to 5. To be sure, some of the terms are negligible in one case and some in other. But it is clear

that *the complete linearized equations can be applied whenever either or both of the conditions* (51) *is not satisfied*, since we are retaining all the necessary terms. The system of linear equations is the following,

$$\left.\begin{aligned} &\phi'_{tt} + 2\phi'_{xt} + (1 - M_1^{-2})\phi'_{xx} - M_1^{-2}\phi'_{yy} = 0 \\ &y = 0, \quad 0 < x < b; \quad \phi'_y = h_x + h_t \\ &C_p = 2(\phi'_x + \phi'_t) \end{aligned}\right\} \tag{52}$$

TABLE 1.

Classification of various cases. Arguments in function for pressure coefficient are the similarity parameters

	$k = 0(1)$			$1 \ll k$	
	Case 1	Case 2	Case 3	Case 4	Case 5
1	$1 - \frac{1}{M_\infty^2} = 0(\delta^{2/3})$ $k = 0(\delta^{2/3})$ $\delta \ll 1$	$\lvert 1 - M_\infty \rvert \gg \delta^{2/3}$ $k = 0\left(1 - \frac{1}{M_\infty^2}\right)$ $\delta \ll 1,\ \delta M_\infty \ll 1$	$k \gg \delta^{2/3}$ $k \gg \left\lvert 1 - \frac{1}{M_\infty^2} \right\rvert$ $\delta \ll 1,\ \delta M_\infty \ll 1$	$k^{-1} = 0(M_\infty)$ $k\delta M_\infty \ll 1$ $k\delta \ll 1$	$kM_\infty \ll 1$ $k\delta \ll 1$
2	Eq. (35)	Eq. (39)	Eq. (42)	Eq. (47)	Eq. (50)
3	$C_p/\delta^{2/3}(\gamma+1)^{1/3}$ $= F\left[\frac{k}{[(\gamma+1)\delta]^{2/3}}, \frac{M_\infty - 1}{[(\gamma+1)\delta]^{2/3}}\right]$	$C_p\sqrt{\lvert 1 - M_\infty^2 \rvert}/\delta$ $= F\left(k, 1 - \frac{1}{M_\infty^2}\right)$	$\sqrt{M_\infty}C_p/\delta$ $= F(k)$	$C_p/\delta k^2$ $= F(kM_\infty)$	$C_p/\delta k^2$ $= F(kM_\infty)$
4	Transonic	subsonic supersonic	oscillating terms dominating		quasi-stationary incompressible
5	Non-linear	linear			

For example, when an airfoil is accelerated through the sonic speed with a sufficiently high acceleration, k can be made to be much larger than $\delta^{2/3}$ and the linear theory applies. The physical interpretation is the following. The breakdown of the linearization is due to the accumulation of disturbances in a steady flow at transonic speeds. On the other hand, if there is a high enough rate of time variation, such disturbances have no chance to build up, and the linearized treatment is justified.

6. Influence of wind tunnel walls. If the airfoil is placed in a two-dimensional wind tunnel, there are further conditions of vanishing vertical velocity at the

walls $y = H_1$ and $y = H_2$ (reduced to non-dimensional form by dividing by the chord length). In order that one may have similarity, there are therefore the additional conditions of identical

$$\eta_1 = \lambda H_1\,, \quad \text{and} \quad \eta_2 = \lambda H_2\,.$$

Referring to Eqs. (18) and (19) and the value of ϵ in the individual cases, one can easily show that the parameters for the five cases are respectively

Case 1.	$[(\gamma+1)\delta]^{1/3}H_1$	$[(\gamma+1)\delta]^{1/3}H_2$	
Case 2.	$\lvert 1-M_\infty^2\rvert^{\frac{1}{3}}H_1$	$\lvert 1-M_\infty^2\rvert^{\frac{1}{3}}H_2$	
Case 3.	$\sqrt{2kM_\infty}H_1$	$\sqrt{2kM_\infty}H_2$	(53)
Case 4.	$kM_\infty H_1$	$kM_\infty H_2$	
Case 5.	H_1	H_2	

7. The Case of Large Pressure Disturbance. In all the above cases, the restrictions

$$M_\infty \ll 1/\delta \quad \text{and} \quad M_\infty \ll 1/k\delta \quad (k \gg 1) \tag{54}$$

are implied. Thus, if either or both of these conditions are not satisfied, the linearized theory breaks down. This can happen when $M_\infty \gg 1$, and is indeed the hypersonic range with non-linear equations. In such cases, even though the velocity perturbations may be small (i.e., condition (13) satisfied), the conditions (13A) for small pressure perturbation cannot be all satisfied. (For if they are, we will go back to one of the linearized cases discussed above); that is, this case cannot be treated properly by a theory of *small perturbations*. Thus, the expansion of the formula (12) for the pressure coefficient cannot be carried out. One can merely write

$$C_p = \frac{2}{\gamma M_1^2}\left\{[1-(\gamma-1)\epsilon M_1^2(kf_\tau + f_\xi + \tfrac{1}{2}\epsilon\lambda^2 f_\eta^2)]^{\gamma/(\gamma-1)} - 1\right\} \tag{55}$$

Indeed, the reduction of the fundamental equation (14) is very limited. One can show that it simplifies to

$$k^2 f_{\tau\tau} + 2kf_{\xi\tau} + f_{\xi\xi} = \epsilon\lambda^2\left\{\left[\frac{1}{\epsilon M_1^2} - (\gamma-1)(kf_\tau + f_\xi) - \tfrac{1}{2}(\gamma+1)\epsilon\lambda^2 f_\eta^2\right]f_{\eta\eta} - 2f_\eta(kf_{\tau\eta} + f_{\xi\eta})\right\} \tag{56}$$

To clarify the problem further, we again distinguish between the cases

$$\text{(A)}\quad k = 0(1), \qquad \text{(B)}\quad k \gg 1$$

Case A. Hypersonic Flow. In this case, we have (cf. (19) and (54))

$$\epsilon\lambda = \delta, \text{ and } 1/M_\infty\delta = 0(1) \tag{57}$$

It turns out that in terms of the parameters

$$\alpha = 1/M_\infty\delta = 0(1) \text{ and } \beta = \epsilon\lambda^2, \tag{58}$$

one can rewrite (55) and (56) as

$$\begin{aligned} C_p &= \frac{2}{\gamma M_\infty^2}\left\{\left[1 - (\gamma - 1)\frac{\alpha^2}{\beta}\left(kf_\tau + f_\xi + \frac{\beta}{2}f_\eta^2\right)\right]^{\gamma/(\gamma-1)} - 1\right\} \\ k^2 f_{\tau\tau} &+ 2kf_{\xi\tau} + f_{\xi\xi} + 2\beta f_\eta(kf_{\eta\tau} + f_{\xi\eta}) \\ &= \beta^2(\alpha^2 - \tfrac{1}{2}(\gamma + 1)f_\eta^2) - (\gamma - 1)\beta(kf_\tau + f_\xi) \end{aligned} \tag{59}$$

It is thus clear that β must be of the order of unity. Since it is at our disposal, one may put

$$\beta = 1. \tag{60}$$

Thus, the final equations for $k = 0(1)$ are

$$\left.\begin{aligned} k^2 f_{\tau\tau} &+ 2kf_{\xi\tau} + f_{\xi\xi} + 2f_\eta(kf_{\eta\tau} + f_{\xi\eta}) \\ &= \{(\alpha^2 - \tfrac{1}{2}(\gamma + 1)f_\eta^2) - (\gamma - 1)(kf_\tau + f_\xi)\}f_{\eta\eta} \\ \eta &= 0, 0 < \xi < 1; \qquad f_\eta = g_\xi + kg_\tau \\ C_p &= \frac{2}{\gamma M_\infty^2}\{[1 - (\gamma - 1)\alpha(kf_\tau + f_\xi + \tfrac{1}{2}f_\eta^2)]^{\gamma/(\gamma-1)} - 1\} \end{aligned}\right\} \tag{61}$$

Case B. Highly Oscillatory Flow. In the case $k \gg 1$, we have (cf. (20) and (54))

$$\epsilon\lambda = k\delta, \text{ and } 1/M_\infty\delta k = 0(1) \tag{62}$$

A similar argument with

$$\alpha' = 1/k\delta M_\infty = 0(1), \beta' = \epsilon\lambda^2 k^{-1} = 1 \tag{63}$$

leads to

$$\left.\begin{aligned} f_{\tau\tau} &+ 2f_\eta f_{\eta\tau} = \{\alpha'^2 - \tfrac{1}{2}(\gamma + 1)f_\eta^2 - (\gamma - 1)f_\tau\}f_{\eta\eta} \\ \eta &= 0, 0 < \xi < 1, \qquad f_\eta = g_\xi \\ C_p &= \frac{2}{\gamma M_\infty^2}\{[1 - (\gamma - 1)\alpha'(f_\tau + \tfrac{1}{2}f_\eta^2)]^{\gamma/(\gamma-1)} - 1\} \end{aligned}\right\} \tag{64}$$

Thus, when $k = 0(1)$, there are the similarity parameters

$$k, \alpha = 1/M_\infty\delta, \text{ and } \gamma;$$

when $k \ll 1$, the problem is quasi-stationary, and the parameter k disappears. In the highly oscillating case, the similarity parameters are

$$\alpha' = 1/M_\infty\delta k \text{ and } \gamma.$$

8. The Transonic case with $M_1 = 1$. If the reference velocity is taken to be the critical velocity, so that $M_1 = 1$, there is an apparent simplification, because the Mach number no longer appears explicitly in the equation (15). However, the boundary condition at infinity introduces it back again. In fact, at infinity,

$$\epsilon f_\xi = 2(M_\infty - 1)/(\gamma + 1),$$

and this introduces a relation between the order of magnitude of ϵ and $(M_\infty - 1)$. Arguments along the above lines lead to the following results.

$$\left.\begin{aligned} &(2k/\delta^{2/3})f_{t\tau} + f_t f_{t\xi} - f_{\eta\eta} = 0 \\ &\eta = 0,\ 0 < \xi < 1; \qquad f_\eta = g_\xi \\ &\xi = -\infty; \qquad f_t = 2\frac{M_\infty - 1}{[(\gamma+1)\delta]^{2/3}}, \qquad f_\eta = 0 \\ &C_p = 2\delta^{2/3}(\gamma+1)^{-1/3} f_t \\ &C_{p\infty} = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = 2\delta^{2/3} f_t - 4(1 - M_\infty)/(\gamma+1) \end{aligned}\right\} \tag{65}$$

and this is readily shown to be equivalent to the contents of (35).

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NOTE ON THE USE OF ALMOST PERIODIC FUNCTIONS IN THE SOLUTION OF CERTAIN BOUNDARY VALUE PROBLEMS[1]

By Ida Roettinger

1. Introduction. This note concerns *linear transformations* of the type $T\{F(x)\} = \int_0^\pi F(x)\varphi_{k_n}(x)dx = f(k_n)$, $n = 1, 2, \cdots$, where $F(x)$ belongs to $L^2(0,\pi)$ and

$$\{\varphi_{k_n}(x)\} = \{A(k_n)\cos k_n x + B(k_n)\sin k_n x\} \tag{1}$$

$A(k_n) = -(k_n a_{12} + a_{13}\sin k_n\pi + a_{14}k_n\cos k_n\pi)$,

$B(k_n) = -a_{11} + a_{13}\cos k_n\pi - a_{14}k_n\sin k_n\pi$,

[compare [6][2] p. 300] is the set of characteristic functions of the Sturm-Liouville problem

$$\begin{gathered} y''(x) + k^2 y(x) = 0, \qquad L_i(y) = \textstyle\sum_{j=1}^4 a_{ij}u_j = 0, \qquad i = 1, 2; \\ u_1 = y(0), \qquad u_2 = y'(0), \qquad u_3 = y(\pi), \qquad u_4 = y'(\pi). \end{gathered} \tag{2}$$

The a_{ij} are constants, the interval $(0, \pi)$ is chosen for convenience and without loss of generality. It is assumed that the L_i are linearly independent and that $a_{11}a_{22} - a_{12}a_{21} = a_{13}a_{24} - a_{14}a_{23}$. This last condition is to guarantee real characteristic values (see [3] vol. 1, p. 352). There is a countable infinity of characteristic values which can be numbered as k_n, $n = 1, 2, \cdots$, in such a way that $k_n < n < k_{n+1}$. Throughout, functions and their transforms are denoted by corresponding capital and small letters. In case the characteristic functions are sines or cosines only, the notations S and C respectively are used instead of T.

According to the Riesz-Fischer theorem (see [4] p. 98) the transformation $T\{F(x)\}$ has a *unique inverse transformation* $T^{-1}\{f(k_n)\}$, almost everywhere, in the form of a Sturm-Liouville series

$$T^{-1}\{f(k_n)\} = \text{l.i.m.}_{m\to\infty} \sum_{n=1}^{m} N(k_n)f(k_n)\varphi_{k_n}(x), \qquad \text{in } (0, \pi), \tag{3}$$

where $N^{\frac{1}{2}}(k_n)$ denotes the normalization factor of the complete set of orthogonal functions $\{\varphi_{k_n}(x)\}$. In other words, there is a one to one mapping of the functions $F(x)$ of $L^2(0, \pi)$ and the sequences $f(k_n)$, $n = 1, 2, \cdots$.

The transformation T is useful in finding quickly a *formal solution* of certain boundary value problems. A detailed discussion of the transformation T as well as the type of boundary value problems to which this operational method is applicable are given in [6].

The purpose of this note is to give relations (see theorems 6 and 7 below) which help in finding *solutions with intrinsic character* (i.e. closed form solutions, com-

[1] See abstract in Bull. Am. Math. Soc. **51**, p. 67.
The content of this note is part of the author's doctoral dissertation.

[2] The numbers in brackets refer to the bibliography.

pare [6] Secs. 6, 9, 10, 11). The relations are obtained by using almost periodic functions in the form of *"almost periodic extensions"* of a function defined on a finite interval. Some simple examples (see Secs. 4 and 5 below) illustrate the use of these relations.

2. Some definitions and theorems from the theory of almost periodic functions. *Definition 1.* A set E of real numbers is said to be *satisfactorily uniform* if there exists a positive number l such that the ratio of the maximum number of terms of E included in an interval of length l to the minimum number is less than 2. (See [1] p. 77.)

Definition 2. Let $F(x)$, $-\infty < x < \infty$, be a real function. The upper and the lower limit (if they exist) of $(1/X)\int_0^X F(x)dx$ as $X \to \infty$ is called the *upper* and the *lower mean value* of $F(x)$, and is denoted by $\bar{M}\{F(x)\}$, $\underline{M}\{F(x)\}$ respectively. In case $\bar{M} = \underline{M}$ the notation M is used. (For more detail see [1] p. 12 and p. 69.)

Definition 3. A function $F(x)$ of class L^2 is called B^2 *almost periodic* ($B^2.a.p.$), if to any $\epsilon > 0$ corresponds a satisfactorily uniform set of numbers $\cdots \tau_{-2} < \tau_{-1} < \tau_0 < \tau_1 < \tau_2 \cdots$ such that for each i $\bar{M}_x\{|F(x+\tau_i) - F(x)|^2\} < \epsilon^2$, and that for every $c > 0$ $\bar{M}_x\bar{M}_i(1/c)\int_x^{x+c} |F(x+\tau_i) - F(x)|^2 dx < \epsilon^2$. The suffixes x and i indicate with respect to which variable the mean value is to be taken. The numbers τ_i are called B^2 *translation numbers* of the function $F(x)$ belonging to ϵ (see [1] pp. 69 and 78).

Notation. By a *u.a.p. function* is meant a uniformly almost periodic function in the sense of Bohr (see [1] p. 2 and [2] p. 28).

Theorem 1. If $F(x)$ is $B^2.a.p.$ then $M\{F(x)\exp - i\lambda x\}$ exists for all real values of λ and differs from naught for at most an enumerable set of λ. Denote these values of λ by $\lambda_1, \lambda_2, \cdots$, and write $M\{F(x)\exp - i\lambda_n x\} = A_n$ for all n. (See [1] p. 18 and p. 104.)

Definition 4. The numbers λ_n in theorem 1 are called the *Fourier exponents* and the numbers A_n, $n = 1, 2, \cdots$, the *Fourier coefficients* of the function $F(x)$. The formal series $\sum A_n \exp i\lambda_n x$ is called the *Fourier series* of the function $F(x)$ and is written as $F(x) \sim \sum A_n \exp i\lambda_n x$ (see [1] p. 18 and p. 104).

Trigonometric form of Fourier series. If $F(x)$ is real in definition 4 and $A_n = a_n - ib_n$, then

$$F(x) \sim \sum a_n \cos \lambda_n x + b_n \sin \lambda_n x, \tag{4}$$

where

$$a_n = M\{F(x)\cos \lambda_n x\}, \qquad b_n = M\{F(x)\sin \lambda_n x\}.$$

Lemma 1. If $F(x)$ is real and almost periodic, then $F(x)$ is even, (odd), if and only if $b_n = 0$, ($a_n = 0$), so that

$$F(x) \sim \sum a_n \cos \lambda_n x, \qquad (F(x) \sim \sum b_n \sin \lambda_n x).$$

Lemma 2. Let $F(x)$ be almost periodic, then $F(x)$ is periodic if and only if its Fourier exponents are integral multiples of some fixed number.

Theorem 2. (Riesz-Fischer). To any series $\sum A_n \exp i\lambda_n x$ for which $\sum |A_n|^2$ converges, corresponds a $B^2.a.p.$ function having this series as its Fourier series and conversely (see [1] pp. 109–110).

Theorem 3. If $F(x)$ and $G(x)$ are $B^2.a.p.$ functions, then

$$H(x) = M_t\{F(x - t)G(t)\}$$

is a $u.a.p.$ function.

Remark. It is the intrinsic character of the $B^2.a.p.$ functions as defined in definition 3, and theorem 2, which the reader should bear in mind in the sections to follow.

3. Almost periodic extensions. In connection with the finite $\sin nx$, $\cos nx$ and the $\sin (n - \frac{1}{2})x$, $\cos (n - \frac{1}{2})x$, $n = 1, 2, \cdots$ transformations, periodic and antiperiodic extensions respectively of a function defined on a finite interval are introduced in order to prove certain theorems (see [6] Secs. 5 and 7). These extensions have their origin in the fact that a series expansion in one of the above four characteristic functions represents a periodic or antiperiodic function respectively on $-\infty < x < \infty$. That the expansion is a periodic or antiperiodic function is due to the fact that these characteristic functions have a common period or antiperiod respectively. In generalizing the above mentioned theorems to certain special cases of the T transformation, where $k_n \neq n$, $k_n \neq n - \frac{1}{2}$, $n = 1, 2, \cdots$, the question arises; what kind of function does the series (3) represent on $-\infty < x < \infty$?

The Fourier exponents, i.e. the characteristic values of (2), are not integral multiples of a fixed number (see theorem 4 below), in other words the characteristic functions (1) have no common period. According to theorem 2 and lemma 2 the series (3) represents an almost periodic function.

Theorem 4. There exists no real number $q \neq 0$ such that k_n/q is an integer for all $n = 1, 2, \cdots$.

The assumption that all numbers k_n are multiples of the same number q leads to a contradiction. This follows from the fact that $\lim_{n\to\infty} (k_n - n) = 0$. The details of the proof can be left to the reader.

Definition 5. A function $\bar{F}(x)$, $-\infty < x < \infty$, is said to be a $B^2.a.p.$ *extension* of a function $F(x)$ belonging to $L^2(0, \pi)$, if $\bar{F}(x)$ is $B^2.a.p.$ and $\bar{F}(x) = F(x)$ in $(0, \pi)$. In particular $\bar{F}_e(x)$, $(\bar{F}_0(x))$, is called an even, (odd), $B^2.a.p.$ extension if $\bar{F}(x)$ is even, (odd).

Theorem 5. Let $F(x)$ belong to $L^2(0, \pi)$, then there exists one and only one $B^2.a.p.$ extension $\bar{F}(x)$, $-\infty < x < \infty$, of $F(x)$ such that

$$\bar{F}(x) \text{ has } k_n, \qquad n = 1, 2, \cdots, \text{ as Fourier exponents} \tag{5}$$

and its corresponding Fourier series $\bar{F}(x) \sim \sum a_n \cos k_n x + b_n \sin k_n x$, $-\infty < x < \infty$, has the property

$$a_n/A(k_n) = b_n/B(k_n), \qquad A(k_n) \text{ and } B(k_n) \text{ as in (1)}, \tag{6}$$

namely

$$\bar{F}(x) \sim \sum N(k_n)f(k_n)\varphi_{k_n}(x) \text{ in } -\infty < x < \infty.$$

Proof. According to theorem 2 the series $\sum N(k_n)f(k_n)\varphi_{k_n}(x)$, $-\infty < x < \infty$, corresponds to a B^2.*a.p.* function, and any B^2.*a.p.* function satisfying (5) and (6) has a uniquely determined Fourier series of this form. According to the uniqueness theorem for expansions in the orthogonal functions $\varphi_{k_n}(x)$ on the interval $(0, \pi)$, there is one and only one such representation of $F(x)$ on this interval; hence this series corresponds to the only B^2.*a.p.* extension $\bar{F}(x)$ of $F(x)$ satisfying (5) and (6).

4. A theorem and a vibrating string problem. If $F(x)$ belongs to $L^2(0, \pi)$ and $\bar{F}(x)$, $-\infty < x < \infty$, is the B^2.*a.p.* extension of $F(x)$ satisfying (5) and (6) then it follows from (4) that

$$\begin{aligned} N(k_n)f(k_n)A(k_n) &= M\{\bar{F}(x)\cos k_n x\}, \\ N(k_n)f(k_n)B(k_n) &= M\{\bar{F}(x)\sin k_n x\}. \end{aligned} \tag{7}$$

Theorem 6. If $F(x)$, $-\infty < x < \infty$, denotes the B^2.*a.p.* extension satisfying (5) and (6) of the function $F(x)$ belonging to $L^2(0, \pi)$ and if a is any constant, then

$$T^{-1}\{\cos k_n a\, f(k_n)\} = \tfrac{1}{2}[\bar{F}(x-a) + \bar{F}(x+a)]. \tag{8}$$

Proof. Since $F(x) = \bar{F}(x)$ in $(0, \pi)$, it follows from (7) that

$$T^{-1}\{\cos k_n a\, f(k_n)\} = \operatorname*{l.i.m.}_{m\to\infty} \sum_{n=1}^{m} M\{\bar{F}(x)\cos k_n a \cos k_n x\}\cos k_n x + M\{\bar{F}(x)\cos k_n a \sin k_n x\}\sin k_n x.$$

Now,

$$\begin{aligned} M\{\bar{F}(x)\cos k_n a\cos k_n x\} &= \tfrac{1}{2}M\{\bar{F}(x)\cos k_n(x+a) + \cos k_n(x-a)\} \\ &= \tfrac{1}{2}M\{[\bar{F}(x-a) + \bar{F}(x+a)]\cos k_n x\}, \end{aligned}$$

similarly

$$M\{\bar{F}(x)\cos k_n a\sin k_n x\} = \tfrac{1}{2}M\{[\bar{F}(x-a) + \bar{F}(x+a)]\sin k_n x\},$$

so that

$$\begin{aligned} T^{-1}\{\cos k_n a\, f(k_n)\} &= \operatorname*{l.i.m.}_{m\to\infty}\sum_{n=1}^{m} \tfrac{1}{2}M\{[\bar{F}(x-a) + \bar{F}(x+a)]\cos k_n x\}\cos k_n x \\ &\quad + \tfrac{1}{2}M\{[\bar{F}(x-a) + \bar{F}(x+a)]\sin k_n x\}\sin k_n x \\ &= \tfrac{1}{2}[\bar{F}(x-a) + \bar{F}(x+a)]. \end{aligned}$$

Corollary. Let $F(x)$ belong to $L^2(0, \pi)$. If $\bar{F}_e(x)$, $(\bar{F}_0(x))$, $-\infty < x < \infty$, denotes the even, (odd), B^2.*a.p.* extension of $F(x)$ with Fourier exponents k_n and if a is any constant, then

$$C^{-1}\{\cos k_n a\, f_c(k_n)\} = \tfrac{1}{2}[\bar{F}_e(x-a) + \bar{F}_e(x+a)], \tag{9}$$

$$S^{-1}\{\cos k_n a\, f_s(k_n)\} = \tfrac{1}{2}[\bar{F}_0(x-a) + F_0(x+a)], \tag{10}$$

where

$$C\{F(x)\} = \int_0^\pi F(x) \cos k_n x dx = f_c(k_n),$$

$$S\{F(x)\} = \int_0^\pi F(x) \sin k_n x dx = f_s(k_n), \qquad n = 1, 2, \cdots;$$

k_n are the characteristic values of (2) where certain a_{ij} respectively are zero.

A vibrating string problem. The closed form solution below of the following simple boundary value problem is meant to illustrate the use of theorem 6.

$$\frac{\partial^2 Y(x,t)}{\partial t^2} = \frac{\partial^2 Y(x,t)}{\partial x^2} + a, \qquad x \text{ in } (0,\pi), \qquad t > 0,$$

$$Y(x, 0+) = 0, \qquad Y_t(x, 0+) = 0,$$

$$Y_x(0+, t) = a\pi + b, \qquad Y_x(\pi -, t) + hY(\pi -, t) = b,$$

a, b, h are given non zero constants.

$$\text{Let } C\{F(x)\} = \int_0^\pi F(x) \cos k_n x dx = f_c(k_n), \qquad n = 1, 2, \cdots, \tag{11}$$

where $\{\cos k_n x\}$ are the characteristic functions of $y''(x) + k^2 y(x) = 0, y'(0) = 0$, $y'(\pi) + hy(\pi) = 0$; that is, k_n are the roots of $\cot k\pi = k/h$, $k > 0$. If $F(x)$ and $F'(x)$ are continuous and $F''(x)$ belongs to $L^2(0, \pi)$, then

$$C\{F''(x)\} = -k_n^2 f_c(k_n) - F'(0) + \cos k_n \pi [F'(\pi) + hF(\pi)], \tag{12}$$

see [6] equation (39). This last formula applied to the above problem yields the following transformed problem.

$$\frac{d^2 y_c(k_n, t)}{dt^2} = -k_n^2 y_c(k_n, t) - (a\pi + b) + b \cos k_n \pi + aC\{1\},$$

$$y_c(k_n, 0+) = 0, \qquad dy_c(k_n, 0+)/dt = 0.$$

The solution of the transformed problem is

$$y_c(k_n, t) = (1 - \cos k_n t) k_n^{-2} [-(a\pi + b) + b \cos k_n \pi + aC\{1\}].$$

Now, $C\{1\} = k_n^{-1} \sin k_n \pi$,

$$C\{x\} = k_n^{-1} \pi \sin k_n \pi + (-1 + \cos k_n \pi) k_n^{-2},$$

$$C\{x^2\} = k_n^{-1} \pi^2 \sin k_n \pi + k_n^{-2} 2\pi \cos k_n \pi - k_n^{-3} 2 \sin k_n \pi,$$

so that $\quad -(a\pi + b)k_n^{-2} + bk_n^{-2} \cos k_n \pi + ak_n^{-3} \sin k_n \pi =$

$$= C\{-(a/2)x^2 + (a\pi + b)x - (a/2)\pi^2 - b\pi\} = C\{F(x)\}.$$

Thus

$$y_c(k_n, t) = C\{F(x)\} - \cos k_n t \, C\{F(x)\} \text{ and by (9)}$$

$$Y(x, t) = F(x) - \tfrac{1}{2}[\bar{F}_e(x - t) + \bar{F}_e(x + t)], \tag{13}$$

where $\bar{F}_e(x)$ denotes the even (in this case) *u.a.p.* extension with Fourier exponents k_n of the function

$$F(x) = -(a/2)x^2 + (a\pi + b)x - (a/2)\pi^2 - b\pi, \qquad 0 < x < \pi.$$

The closed form solution (13) is of intrinsic character and thus gives an insight into the behaviour of the solution.

Remark. In solving practical problems of this type it is worth while keeping in mind that the roots k_n of (2) approach n, in some cases $n - \frac{1}{2}$, $n = 1, 2, \cdots$, asymptotically. For instance, if in the above problem $|h|$ is small, k_n is approximately n, in which case $\bar{F}_e(x)$ in (13) is approximately the even periodic extension with period 2π of the function $F(x)$ defined on $(0, \pi)$.

5. A convolution theorem and a problem in heat conduction. The following theorem gives an expression for the inverse of the product of two C-transforms.

Theorem 7. Let $F(x)$ and $G(x)$ belong to $L^2(0, \pi)$. If $\bar{F}_e(x)$ and $\bar{G}_e(x)$, $-\infty < x < \infty$, denote the even B^2.*a.p.* extensions with Fourier exponents k_n of the functions $F(x)$ and $G(x)$ respectively, and if $H(x) = M_t\{\bar{F}_e(x - t)\bar{G}_e(t)\}$ and $\bar{H}(x) = \int_0^x H(v)dv$, then

$$C^{-1}\{f_e(k_n)g_e(k_n)\} = (\pi/2)H(x) - (1/8)[\bar{H}(x + 2\pi) - \bar{H}(x - 2\pi)] \quad (14)$$

Proof. $C^{-1}\{f_e(k_n)g_e(k_n)\} = \underset{m\to\infty}{\text{l.i.m.}} \sum_{n=1}^{m} N(k_n)f_e(k_n)g_e(k_n) \cos k_n x$. According to (7)

$$N(k_n)f_e(k_n)g_e(k_n) = [1/\dot{N}(k_n)]M\{\bar{F}_e(x) \cos k_n x\}M\{\bar{G}_e(x) \cos k_n x\}.$$

And

$$M\{\bar{F}_e(x) \cos k_n x\}M\{\bar{G}_e(x) \cos k_n x\} = M_x\{M_y\{\bar{F}_e(x)\bar{G}_e(y) \cos k_n x \cos k_n y\}\}$$
$$= \tfrac{1}{2}[M_x\{M_y\{\bar{F}_e(x)\bar{G}_e(y)[\cos k_n(x + y) + \cos k_n(x - y)]\}\}] = M\{H(x) \cos k_n x\},$$

which follows from the substitution $x + y = u$, $y = v$ in the first term and $x - y = u$, $y = -v$ in the second term.

Next, $1/N(k_n) = \int_0^\pi \cos^2 k_n x dx = (\pi/2) + (4k_n)^{-1} \sin 2k_n\pi$. Now,

$$k_n^{-1}M\{H(x) \cos k_n x\} = \lim_{x\to\infty} (1/X)[k_n^{-1} \cos k_n x\bar{H}(x)]_0^X = M\{\bar{H}(x) \sin k_n x\}.$$

Since $\bar{H}(x)$ is bounded the first term vanishes, so that

$$k_n^{-1}M\{H(x) \cos k_n x\} = M\{\bar{H}(x) \sin k_n x\}.$$

Similarly as in the proof to theorem 6 it follows, that

$$\sin 2k_n\pi M\{\bar{H}(x) \sin k_n x\} = \tfrac{1}{2}M\{[\bar{H}(x + 2\pi) - \bar{H}(x - 2\pi)] \cos k_n x\},$$

so that

$$C^{-1}\{f_e(k_n)g_e(k_n)\} = (\pi/2)H(x) + (1/8)[\bar{H}(x + 2\pi) - \bar{H}(x - 2\pi)].$$

Auxiliary problem. The solution of this problem is of use in giving a closed form solution of the heat conduction problem below.

$$\partial\Psi/\partial t = \partial^2\Psi/\partial x^2, \qquad x \text{ in } (0, \pi), \qquad t > 0,$$

$$\Psi_x(0+, t) = -1, \qquad \Psi_x(\pi-, t) + h\Psi(\pi-, t) = 0, \qquad \Psi(x, 0+) = 0.$$

Relation (12) leads to the following transformed problem.

$$d\psi_c/dt = -k_n^2\psi_c + 1, \qquad \psi_c(k_n, 0+) = 0.$$

Hence

$$\psi_c(k_n, t) = k_n^{-2}(1 - \exp - k_n^2 t),$$

and $\Psi(x, t) = \underset{m\to\infty}{\text{l.i.m.}} \sum_{n=1}^{m} 4k_n^{-1}(2k_n\pi + \sin 2k_n\pi)^{-1}(1 - \exp - k_n^2 t)\cos k_n x.$

A heat conduction problem. The closed form solution below of the following simple boundary value problem is meant to illustrate the use of theorem 7.

$$\partial U/\partial t = A(t)\partial^2 U/\partial x^2, \qquad x \text{ in } (0, \pi), \qquad t > 0,$$

$$U_x(0+, t) = -1, \qquad U_x(\pi-, t) + hU(\pi-, t) = 0, \qquad U(x, 0+) = F(x).$$

$A(t) > 0$, $F(x)$ and h are given. According to (12) the following transformed problem is obtained.

$$du_c(k_n, t)/dt = A(t)[-k_n^2 u_c(k_n, t) + 1],$$

$$u_c(k_n, 0+) = f_c(k_n),$$

thus $u_c(k_n, t) = f_c(k_n)\psi_c(k_n, \alpha(t))$, where Ψ denotes the solution of the auxiliary problem and $\alpha(t) = \int_0^t A(v)dv$.

By theorem 7 the solution $U(x, t)$ can be written as

$$U(x, t) = (\pi/2)H(x) + (1/8)[\bar{H}(x + 2\pi) - \bar{H}(x - 2\pi)], \tag{15}$$

where $H(x) = M_v\{\bar{F}_e(x - v)\Psi_e(v, \alpha(t))\}$, Ψ_e denoting the even $B^2.a.p.$ extension with respect to v of the function ψ.

Remark. The same remark as in Sec. 4 holds here. In case $|h|$ is small k_n is approximately n. In this case (14) can be approximated by $C^{-1}\{f_c(n)g_c(n)\} = \frac{1}{2}\int_{-\pi}^{\pi} F_e(x - v)G_e(v)dv$, where F_e, G_e denote the even periodic extensions with period 2π respectively of $F(x)$ and $G(x)$ defined on $(0, \pi)$. And (15) is thus approximately given by $U(x, t) = \frac{1}{2}\int_{-\pi}^{\pi} F_e(x - v)\Psi_e(v, \alpha(t))dv$, where Ψ_e denotes the even periodic extension with period 2π with respect to v of the function $\Psi(v, \alpha(t))$, $0 < v < \pi$, and
$\Psi(x, t) = (t/\pi) + (\pi - x)^2/2\pi - (\pi/6) - (2/\pi)\sum_1^\infty n^{-2}\exp(-n^2 t)\cos nx$, x in $(0, \pi)$.

BIBLIOGRAPHY

1. A. S. Besicovitch, Almost periodic functions, Cambridge University Press 1932.
2. H. Bohr, Fastperiodische Funktionen, Springer, Berlin 1932.
3. Frank-v. Mises, Differentialgleichungen der Physik, 2. vermehrte Auflage, New York 1943.
4. S. Kaczmarz and H. Steinhaus, Theorie der Orthogonalreihen, Warsaw 1935.
5. C. F. Muckenhoupt, Almost periodic functions and vibrating systems, Journal of Math. and Phys. **8**, 163–199.
6. I. Roettinger, A generalization of the finite Fourier transformation and applications, Quarterly of applied math. **5**, 298–319, consult bibliography therein.

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CORRECTIONS TO THE PAPER "STRESSES AND SMALL DISPLACEMENTS OF SHALLOW SPHERICAL SHELLS. II"

By Eric Reissner

In this paper which appeared in vol. 25, p. 279 (1946) the following corrections should be made: Eqs. (42) and (43) should read

$$M_{rr} = \frac{D}{l^2}\left\{C_1\left[\operatorname{bei}\frac{r}{l} + (1-\nu)\frac{\operatorname{ber}' r/l}{r/l}\right] - C_2\left[\operatorname{ber}\frac{r}{l} - (1-\nu)\frac{\operatorname{bei}' r/l}{r/l}\right]\right.$$
$$\left. + C_3\left[\operatorname{kei}\frac{r}{l} + (1-\nu)\frac{\operatorname{ker}' r/l}{r/l}\right] - C_4\left[\operatorname{ker}\frac{r}{l} - (1-\nu)\frac{\operatorname{kei}' r/l}{r/l}\right]\right\} \tag{42}$$

$$M_{\theta\theta} = \frac{D}{l^2}\left\{C_1\left[\nu\operatorname{bei}\frac{r}{l} - (1-\nu)\frac{\operatorname{ber}' r/l}{r/l}\right] - C_2\left[\nu\operatorname{ber}\frac{r}{l} + (1-\nu)\frac{\operatorname{bei}' r/l}{r/l}\right]\right.$$
$$\left. + C_3\left[\nu\operatorname{kei}\frac{r}{l} - (1-\nu)\frac{\operatorname{ker}' r/l}{r/l}\right] - C_4\left[\nu\operatorname{ker}\frac{r}{l} + (1-\nu)\frac{\operatorname{kei}' r/l}{r/l}\right]\right\} \tag{43}$$

In Eq. (61b) replace $r \leqq r_p$ by $r_p \leqq r$. In Eqs. (62a, b) replace $(r - \nu^2)$ by $(1 - \nu^2)$. Eqs. (64a, b) should read

$$\mp\frac{2\pi\mu^2 l^2 \sigma_{\theta,B}}{12P} = \begin{cases} -c_1\left(\nu\operatorname{bei}\dfrac{r}{l} - \dfrac{1-\nu}{\gamma/l}\operatorname{ber}'\dfrac{r}{l}\right) + c_2\left(\nu\operatorname{ber}\dfrac{r}{l} + \dfrac{1-\nu}{\gamma/l}\operatorname{ber}'\dfrac{r}{l}\right) & (64a) \\ -c_3\left(\nu\operatorname{kei}\dfrac{r}{l} - \dfrac{1-\nu}{\gamma/l}\operatorname{ker}'\dfrac{r}{l}\right) + c_4\left(\nu\operatorname{ker}\dfrac{r}{l} + \dfrac{1-\nu}{\gamma/l}\operatorname{kei}'\dfrac{r}{l}\right) & (64b) \end{cases}$$

Eqs. (63c), (64c) should read

$$\sigma_{r,B}(0) = \sigma_{\theta,B}(0) = \mp\frac{3(1+\nu)}{\pi'}\frac{P}{l^2}\frac{c_2}{\mu^2}. \qquad (63c),\quad (64c)$$

In Eq. (75f) replace $\frac{1}{2}\lambda$ by $1/2\lambda$. In Table I replace k by κ, and c_1, c_2 by C_1, C_2.

The author is indebted to Mr. J. H. Carr of the U. S. Naval Ordnance Test Station at Inyokern for pointing out to him the majority of the aforementioned corrections.

None of the numerical results obtained in the paper are affected by these corrections.

SOUND RADIATION FROM A FINITE CYLINDER

PIERO GIORGIO BORDONI AND WOLF GROSS

1. Introduction. To calculate sound pressure at every point of the field originated by a vibrating body, whose normal vibration velocity is given on its whole surface, a Neumann's external problem for the wave equation: $(\Delta^2 + k^2)\Phi = 0$ must be solved.

In a previous note [6] one of the authors has shown that it is possible to get an approximate solution of this problem, for a sound source of any shape, by means of spherical harmonical analysis. The process outlined in the paper quoted consists in retaining only a finite number of terms from the expansion of the velocity potential in terms of Legendre functions. Boundary conditions must be then satisfied not on the whole surface, but in a finite number of points, in order to obtain as many equations as arbitrary coefficients in the series.

Only a linear system of independent equations, which are just the same number as the unknowns, must be solved with the above method. This is a much easier problem than the search for the integral satisfying to boundary conditions all over the body's surface, and therefore a numerical solution can always be obtained, for any shape of the source.

The same method has subsequently been improved by the second author, according to the suggestions of Prof. M. Picone. Instead of making the approximate solution to converge to the exact one in a discrete number of points, it has been required that it converge in the average all over the surface.

For this purpose the difference between the given velocity and the velocity of the sound field, has first been squared, and afterwards integrated over the whole surface, rendering this integral a minimum by a suitable choice of the expansion's coefficients.

The chief advantage that the latter method has over the former, is that not only its solutions but also all their derivatives converge uniformly to the exact values, when the number of terms retained from the series tends to infinity. This property can be proved without much difficulty as it is done in the Appendix, and is a very significant one from a physical viewpoint, as the acoustical velocity, involved in boundary conditions, is exactly one of the potential derivatives.

In the present note the last method has been employed to calculate the power radiated, by a rigid cylinder with one vibrating basis, (Fig. 1a) and to trace its directional characteristics.

Such a problem, besides its theoretical interest, is also important for technical applications, as the cylindrical source is obviously a schematisation of a moving coil loudspeaker, applied to a back-enclosed baffle (Fig. 1b). It may be seen from the bibliography that not very much is actually known on the behavior of this source; some information about it has been however obtained in previous work by means of a process that could be termed of "physical approximation," which consists in substituting to the true source a rigid sphere with a vibrating cap (Fig. 1c). If the sphere has the same volume as the baffle, and the cap the

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same surface and strength[1] as the loudspeaker's cone, it can be assumed, on physical grounds, that the low-frequency behavior of both sources is similar.

If the spherical source is taken as a first approximation to the back-enclosed loudspeaker, the vibrating cylinder must be considered as a second and closer one. Its contour is indeed much more like the loudspeaker's shape, and it may reasonably be expected that its behavior too be more similar to that of the true source.

A third approximation could eventually be obtained by applying the same analytical process to a cylinder with one concave basis, exactly like in Fig. 1b. Such a refinement would require more troublesome calculations, because the odder the source shape, the larger the number of terms of the series which must

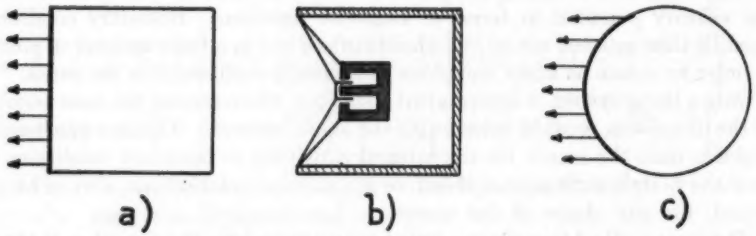

FIG. 1. Sound sources with back-enclosed baffle; a) cylindrical; b) moving coil loudspeaker; c) equivalent spherical source.

be retained. The present note shows however that a further approximation of this kind is not needed, the agreement between spherical and cylindrical source being good enough, at least for technical purposes, and below a certain frequency.

When this frequency is exceeded the agreement becomes poorer, and for very high frequencies the two sources behave in a quite different way. This, however, does not matter very much, as, for such frequencies, computations must be started on an entirely different basis, owing to the loudspeaker's cone breaking up in very complicated modes of vibration.

2. Velocity potential and boundary conditions. In spherical coordinates the velocity potential φ of a sound wave, symmetrical about the polar axis, is given in terms of Legendre functions by the following series:

$$[1] \qquad \varphi = \Phi e^{j\omega t} = (e^{-jkr}/r) \sum_{n=0}^{\infty} (x_n + jy_n) P_n(\cos \vartheta)\, f_n(jkr)$$

where

r, ϑ = spherical symmetrical coordinates;
$k = \omega/c = 2\pi/\lambda$ = phase constant of propagation;
$x_n + jy_n$ = arbitrary coefficient;
$P_n(\cos \vartheta)$ = Legendre function, of order n;
$f_n(jkr)$ = Stokes function, of order n [7].

[1] Integral of the normal vibration velocity, taken all over the surface of the cap or of the diaphragm.

The radial velocity U_r at any point of the field, and the transversal one U_ϑ, which lies on a diametral plain because of symmetry, are given by:

$$[2] \qquad U_r = -\frac{\partial \varphi}{\partial r} = (e^{-jkr}/r^2) \sum_{n=0}^{\infty} (x_n + jy_n) P_n(\cos \vartheta) F_n(jkr)$$

$$[3] \qquad U_\vartheta = -\frac{\partial \varphi}{r \partial \vartheta} = (e^{-jkr}/r^2) \sin \vartheta \sum_{n=0}^{\infty} (x_n + jy_n) P'_n(\cos \vartheta) f_n(jkr).$$

where $P'_n(\cos \theta) = dP_n(\cos \theta)/d\theta$ and where $F_n(jkr)$ is the Stokes function of the second kind, and n^{th} order [7].

If the origin of coordinates is chosen in the center of the cylinder, and its axis is taken as polar axis with $\vartheta = 0$ on the vibrating basis (fig. 2), the velocity

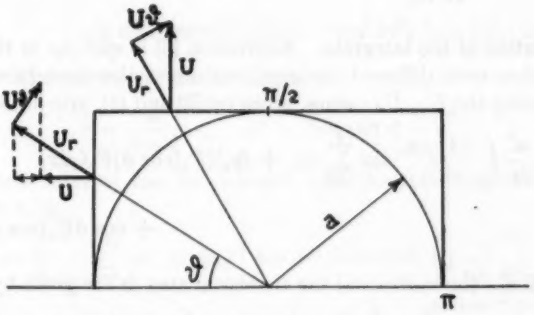

FIG. 2. Components of vibration velocity in a cylindrical source

U normal to the cylinder, can be expressed as a sum of the projected values of U_r, U_ϑ;

$$[4] \qquad \begin{cases} U = U_r \cos \vartheta - U_\vartheta \sin \vartheta & \text{(on the bases)} \\ U = U_r \sin \vartheta + U_\vartheta \cos \vartheta & \text{(on the wall)} \end{cases}$$

If the cylinder's height and diameter are supposed to have the same value a, the radius r is given by:

$$[5] \qquad \begin{cases} r = a/\cos \vartheta, \text{ on the vibrating basis, where } 0 \leq \vartheta \leq \pi/4; \\ r = a/\sin \vartheta, \text{ on the wall, where } \pi/4 \leq \vartheta \leq 3\pi/4; \\ r = -a/\cos \vartheta, \text{ on the standing basis, where } 3\pi/4 \leq \vartheta \leq \pi. \end{cases}$$

Retaining only the first $p + 1$ terms from the expansions [2] and [3], and introducing them into [4], to get the normal velocity, the coefficients $x_n + jy_n$ must be chosen in such a way, as to minimize the integral, taken all over the surface, of the squared difference between given and calculated velocity. The former velocity is zero on the wall and on one basis of the cylinder, whilst on the other basis it has a uniform value, which can be assumed real and equal to $1/a^2$, without any loss in generality. The values that will be obtained, for

the coefficients $x_n + jy_n$ hold also for an arbitrary given velocity U, provided they are multiplied by a^2U.

Denoting with S_0 the wall of the cylinder, with S_1 the vibrating basis, and with S_2 the standing one, the function which must be minimized by a suitable choice of the $x_n + jy_n$, is then;

$$I^{(p)} = \frac{a^2}{2\pi}\int_{S_0} |\, U_r^{(p)} \sin\vartheta + U_\vartheta^{(p)} \cos\vartheta \,|^2\, ds$$

$$+ \frac{a^2}{2\pi}\int_{S_1} \left| U_r^{(p)} \cos\vartheta - U_\vartheta^{(p)} \sin\vartheta - \frac{1}{a^2}\right|^2 ds \tag{6}$$

$$+ \frac{a^2}{2\pi}\int_{S_2} |\, U_r^{(p)} \cos\vartheta - U_\vartheta^{(p)} \sin\vartheta \,|^2\, ds = I_0^{(p)} + I_1^{(p)} + I_2^{(p)}.$$

3. Computation of the integrals. Expression [6] is split up in three integrals, which are taken over different surfaces, and must therefore be evaluated distinctly. Putting the U_r, U_ϑ values, given by [2] and [3], into [6], we get for $I_0^{(p)}$:

$$I_0^{(p)} = \frac{a^2}{2\pi}\int_{S_0} \left| \frac{1}{r^2}\, e^{-jkr} \sin \sum_{n=0}^{p} (x_n + jy_n)[P_n(\cos\vartheta)F_n(jkr)\right.$$

$$\left. + \cos\vartheta P_n'(\cos\vartheta) f_n(jkr)]\right|^2 ds \tag{7}$$

On the wall S_0, the radius and the elemental area ds are given by [5]:

$$r = \frac{a}{\sin\vartheta}; \qquad ds = 2\pi a \frac{a\, d\vartheta}{\sin^2\vartheta} \tag{8}$$

hence the surface integral can be changed into a line integral, for the ϑ variable. Taking for brevity $ka = \alpha$ we obtain:

$$I_0^{(p)} = \int_{\pi/4}^{3\pi/4} \left| \sum_{n=0}^{p} (x_n + jy_n) \left[P_n(\cos\vartheta) F_n\left(j\frac{\alpha}{\sin\vartheta}\right)\right.\right.$$

$$\left.\left. + \cos\vartheta P_n'(\cos\vartheta) f_n\left(j\frac{\alpha}{\sin\vartheta}\right)\right]\right|^2 \sin^4\vartheta\, d\vartheta. \tag{9}$$

Putting $(\pi/2) - \vartheta$ in the place of ϑ, and introducing a new function $\xi_n(\vartheta)$, given by:

$$\xi_n(\vartheta) = P_n(\sin\vartheta) F_n\left(j\frac{\alpha}{\sin\vartheta}\right) + \sin\vartheta P_n'(\sin\vartheta) f_n\left(j\frac{\alpha}{\cos\vartheta}\right) \tag{10}$$

after some algebraic juggling, the following expression can be established for $I_0^{(p)}$:

$$I_0^{(p)} = \sum_{n=0}^{p}\sum_{\nu=0}^{p} (x_n + jy_n)(x_n - jy_n) \int_{-\pi/4}^{+\pi/4} \xi_n(\vartheta)\bar{\xi}_\nu(\vartheta) \cos^4\vartheta\, d\vartheta$$

$$= \sum_{n=0}^{p}\sum_{\nu=0}^{p} A_{n,\nu}(x_n + jy_n)(x_n - jy_n) \tag{11}$$

It may be observed that, $\xi_n(-\theta) = (-1)^n \xi_n(\vartheta)$ and therefore the coefficients $A_{n,\nu}$ have the values:

$$[12] \qquad A_{n,\nu} = \bar{A}_{\nu,n} = \begin{cases} 2\int_0^{\pi/4} \xi_n \xi_\nu \cos^4 \vartheta \, d\vartheta, & \text{when } n+\nu \text{ is even;} \\ 0 & \text{when } n+\nu \text{ is odd.} \end{cases}$$

The above values can be easily calculated, as they are given by integrals of polynomials in $\sin \vartheta$ and $\cos \vartheta$.

Proceeding along the same lines as before, that is putting [2] and [3] into $I_1^{(p)}$ we get:

$$[13] \qquad I_1^{(p)} = \frac{a^2}{2\pi} \int_{S_1} \left| \frac{1}{r^2} e^{-jkr} \sum_{n=0}^{p} (x_n + jy_n) \cdot \left[\cos \vartheta P_n(\cos \vartheta) F_n(jkr) - \sin^2 \vartheta P_n'(\cos \vartheta) f_n(jkr) - \frac{1}{a^2} \right] \right|^2 ds.$$

On the vibrating basis, the first of [5] gives:

$$[14] \qquad r = \frac{a}{\cos \vartheta}; \qquad ds = 2\pi a \tan \vartheta \frac{a \, d\vartheta}{\cos^2 \vartheta}$$

and the surface integral can be changed into the following one, taken from $\vartheta = 0$ to $\vartheta = \pi/4$.

$$[15] \qquad I_1^{(p)} = \int_0^{\pi/4} \left| \sum_{n=0}^{p} (x_n + jy_n) \left[\cos \vartheta P_n(\cos \vartheta) F_n\left(j \frac{\alpha}{\cos \vartheta}\right) - \sin^2 \vartheta P_n'(\cos \vartheta) f_n\left(j \frac{\alpha}{\cos \vartheta}\right) - \frac{e^{j(\alpha/\cos\vartheta)}}{\cos^2 \vartheta} \right] \right|^2 \cos \vartheta \sin \vartheta \, d\vartheta$$

Putting $t = \cos \vartheta$, and introducing the function:

$$[16] \qquad \eta_n(t) = tP_n(t)F_n(j\alpha/t) - (1 - t^2)P_n'(t)f_n(j\alpha/t),$$

the integral [15] becomes:

$$[17] \qquad I_1^{(p)} = \int_{1/\sqrt{2}}^{1} \left| \sum_{n=0}^{p} (x_n + jy_n)\eta_n(t) - \frac{e^{j\alpha/t}}{t^2} \right|^2 t \, dt$$

It is easy to see that the integral $I_2^{(p)}$, taken over the other basis, can be put in a similar form, with the slightly different change of variable $t = -\cos \vartheta$:

$$[18] \qquad I_2^{(p)} = \int_{1/\sqrt{2}}^{1} \left| \sum_{n=0}^{p} (-1)^{n-1}(x_n + jy_n)\eta_n(t) \right|^2 t \, dt$$

As $I_1^{(p)}$ and $I_2^{(p)}$ are both taken from $1/\sqrt{2}$ to 1 they can be evaluated together; after some transformations we get:

$$[19] \qquad I_1^{(p)} + I_2^{(p)} = \sum_{n=0}^{p} \sum_{\nu=0}^{p} (x_n + jy_n)(x_n - jy_n)[1 + (-1)^{n+\nu}] \cdot \int_{1/\sqrt{2}}^{1} \eta_n(t)\eta_\nu(t) t \, dt - \sum_{n=0}^{p} \left[(x_n + jy_n) \int_{1/\sqrt{2}}^{1} \eta_n(t) \frac{e^{-j\alpha/t}}{t} dt + (x_n - jy_n) \int_{1/\sqrt{2}}^{1} \bar{\eta}_n(t) \frac{e^{j\alpha/t}}{t} dt \right] + \int_{1/\sqrt{2}}^{1} \frac{dt}{t^3}$$

Introducing two new parameters, $B_{n,\nu}$, C_n, given by

$$[20] \qquad B_{n,\nu} = \begin{cases} 2\int_{1/\sqrt{2}}^{1} \eta_n(t)\bar{\eta}_\nu(t)t\,dt, & \text{(when } n+\nu \text{ is even)}, \\ 0, & \text{(when } n+\nu \text{ is odd)}, \end{cases}$$

$$[21] \qquad C_n = \int_{1/\sqrt{2}}^{1} \eta_n(t)\,\frac{e^{-j\alpha/t}}{t}\,dt$$

the sum $I_1^{(p)} + I_2^{(p)}$ can be written in the form

$$[22] \qquad I_1^{(p)} + I_2^{(p)} = \sum_{n=0}^{p}\sum_{\nu=0}^{p} B_{n,\nu}(x_n + jy_n)(x_\nu - jy_\nu) \cdot \\ - \sum_{n=0}^{p} [C_n(x_n + jy_n) + \bar{C}_n(x_n - jy_n)] + \tfrac{1}{2}$$

and can be calculated without difficulty, $B_{n,\nu}$ and C_n being given by integrals of polynoms in t.

From [6], [11] and [22] the function $I^{(p)}$ is easily obtained in terms of the same $x_n + jy_n$ which must render it a minimum:

$$[23] \qquad I^{(p)} = \sum_{n=0}^{p}\sum_{\nu=0}^{p} (A_{n,\nu} + B_{n,\nu})(x_n + jy_n)(x_\nu - jy_\nu) \\ - \sum_{n=0}^{p} [C_n(x_n + jy_n) + \bar{C}_n(x_n - jy_n)] + \tfrac{1}{2}$$

3. Evaluation of the coefficients. On differentiating [23] with respect to x_n and y_n, a system of linear equations for the coefficients is obtained

$$[24] \qquad \begin{aligned} \partial I^{(p)}/\partial x_n &\equiv \sum_{\nu=0}^{p} [(A_{n,\nu} + B_{n,\nu})(x_\nu - jy_\nu) \\ &\quad + (\bar{A}_{n,\nu} + \bar{B}_{n,\nu})(x_\nu + jy_\nu)] - C_n - \bar{C}_n = 0, \\ j^{-1}\partial I^{(p)}/\partial y_n &\equiv \sum_{\nu=0}^{p} [(A_{n,\nu} + B_{n,\nu})(x_\nu - jy_\nu) \\ &\quad - (\bar{A}_{n,\nu} + \bar{B}_{n,\nu})(x_\nu + jy_\nu)] - C_n + \bar{C}_n = 0 \\ &\quad (n = 0, 1, 2 \cdots p]. \end{aligned}$$

If we take:

$$[25] \qquad A_{n,\nu} + B_{n,\nu} = \alpha_{n,\nu} + j\beta_{n,\nu}, \qquad C_n = \gamma_n + j\delta_n$$

the preceding system reduces to:

$$[26] \qquad \begin{cases} \sum_{\nu=0}^{p} \alpha_{n,\nu}x_\nu + \sum_{\nu=0}^{p} \beta_{n,\nu}y_\nu = \gamma_n \\ \sum_{\nu=0}^{p} \beta_{n,\nu}x_\nu - \sum_{\nu=0}^{p} \alpha_{n,\nu}y_\nu = \delta_n \end{cases}$$

It must be noted that when $n + \nu$ is odd, $\alpha_{n,\nu} = \beta_{n,\nu} = 0$, as follows from [12] and [20]. Hence equations [26] contain only the terms for which n and ν

are both even or both odd. The system then splits up into two separate systems; the former contains $p + 2$ equation for the $p + 2$ unknowns which have an even index if p is even and vice-versa; the latter has only p equations, and the index must be now even if p is odd.

From the expressions derived for $A_{n,\nu}$, $B_{n,\nu}$, it immediately follows:

$$[27] \qquad \alpha_{n,\nu} = \alpha_{\nu,n}, \qquad \beta_{n,\nu} = -\beta_{\nu,n}, \qquad \beta_{n,n} = 0$$

To evaluate sound radiation accurately enough for technical purposes, not many terms must be retained from the series, as the contribution of successive harmonics to radiation quickly vanishes as their order increases. As it was shown in a previous note (5) four or five terms generally suffice; taking for instance $p = 4$ the boundary equations [26] are then written:

$$[27] \quad \begin{cases} \alpha_{00}x_0 + \alpha_02x_2 + \alpha_{04}x_4 + \beta_{02}y_2 + \beta_{04}y_4 = \gamma_0, \\ \alpha_{02}x_0 + \alpha_{22}x_2 + \alpha_{24}x_4 - \beta_{02}y_0 + \beta_{24}y_4 = \gamma_2, \\ \alpha_{04}x_0 + \alpha_{24}x_2 + \alpha_{44}x_4 - \beta_{04}y_0 - \beta_{24}y_2 = \gamma_4, \\ \beta_{02}x_2 + \beta_{04}x_4 - \alpha_{00}y_0 - \alpha_{02}y_2 - \alpha_{04}y_4 = \delta_0, \\ -\beta_{02}x_0 + \beta_{24}x_4 - \alpha_{02}y_0 - \alpha_{22}y_2 - \alpha_{24}y_4 = \delta_2, \\ -\beta_{04}x_0 - \beta_{24}x_2 - \alpha_{04}y_0 - \alpha_{24}y_2 - \alpha_{44}y_4 = \delta_4, \end{cases}$$

$$[28] \quad \begin{cases} \alpha_{11}x_1 + \alpha_{13}x_3 + \beta_{13}y_3 = \gamma_1, \\ \alpha_{13}x_1 + \alpha_{33}x_3 - \beta_{13}y_1 = \gamma_3, \\ \beta_{13}x_3 - \alpha_{11}y_1 - \alpha_{13}y_3 = \delta_1, \\ -\beta_{13}x_1 - \alpha_{13}y_1 - \alpha_{33}y_3 = \delta_3. \end{cases}$$

To evaluate $\alpha_{n,\nu}$, $\beta_{n,\nu}$ in the above equations the integrals $A_{n,\nu}$, $B_{n,\nu}$ must be computed. Their calculation however does not offer serious difficulties, and both can be expressed in terms of elementary functions. The expression of $A_{n,\nu}$ is of the type:

$$[29] \qquad A_{n,\nu} = \sum_{k=0}^{n+\nu} D_k^{(n+\nu)} \frac{1}{\alpha^n} + E_{n+\nu}\alpha^2 + j(F_{n,\nu}\alpha + G_{n,\nu}/\alpha)$$

where $D_k^{(n+\nu)}$, $E_{n,\nu}$, $F_{n,\nu}$, $G_{n,\nu}$ are numerical coefficients. For the $B_{n,\nu}$ holds a similar equation but with different values of the coefficients.

The computation of γ_n, δ_n is somewhat more complicated, as the integrals C_n cannot be given in terms of elementary functions. It will be remembered that $\eta_n(t)$ is a polynom in t, and therefore the integrals to evaluate are of the following type:

$$[30] \qquad T_s(\alpha) = \int_{1/\sqrt{3}}^{1} t^s e^{-j\alpha/t}\, dt \qquad (s = -1, 0, 1, 2 \cdots)$$

If we suppose $s > 0$, and integrate by parts, we get:

$$[31] \qquad T_s(\alpha) = \left[\frac{t^{s+1}}{s+1}\, e^{-j\alpha/t}\right]_{1/\sqrt{2}}^{1} - \int_{1/\sqrt{2}}^{1} \frac{t^{s+1}}{s+1}\, e^{j\alpha/t}\, \frac{j\alpha}{t^2}\, dt$$

whence a useful recurrence formula can be derived, which reduces the calculation of T_s to that of T_{-1}.

The latter can be expressed by

$$[32] \qquad T_{-1}(\alpha) = \int_{1/\sqrt{2}}^{1} e^{-j(\alpha/t)}\, \frac{dt}{t} = \int_{a}^{a\sqrt{2}} e^{-jx}\, \frac{dx}{x} = \int_{a}^{a\sqrt{2}} \frac{\cos x}{x}\, dx - j \int_{a}^{a\sqrt{2}} \frac{\sin x}{x}\, dx$$

Introducing the known functions,

$$Ci(x) = -\int_{x}^{\infty} \frac{\cos u}{u}\, du, \qquad Si(x) = \int_{x}^{\infty} \frac{\sin u}{u}\, du$$

whose values are tabulated, the integral T_s and therefore the coefficients C_n, γ_n, δ_n can be computed for given α's.

4. Directional characteristics and power radiated. Once the constants $\alpha_{n,\nu}$, $\beta_{n,\nu}$, γ_n, δ_n are evaluated, equations [27], [28] may be solved, and a set of $x_n + jy_n$ found for any given value of $ka = \alpha$. In the present case the following values of α have been chosen:

$$\alpha = 0.5;\ 1.0;\ 2.5;\ 5.0$$

which correspond to the numerical values for $(x_n + jy_n)$ listed in Table I.

It must be noted that it is unnecessary to take values smaller than 0.5 for α, as the behavior of a source does not depend any more on its shape when frequency vanishes. On the other hand values larger than 5.0 are of no use, as the cone vibrates with one nodal circle when the frequency is near to that of its main symmetrical mode, and therefore cannot be any longer thought of as a rigid body.

A simple calculation shows that 5.0 is the largest value which can be safely given to α. Although no general expression has yet been derived for the symmetrical frequencies of a free-edge cone, the fundamental one can be evaluated by means of the approximate formula given by N. W. MacLachlan (1)

$$[33] \qquad f_{1,0} = \frac{\cot\left(\frac{1}{2}\varphi\right)}{2\pi a} \left[\frac{E}{\rho_s}\right]^{\frac{1}{2}}$$

where φ = apical angle, a = radius of the basis, E = Young's modulus, ρ_s = density.

Taking: $\varphi = \pi/2$, $[E/\rho_s]^{\frac{1}{2}} = 1.9 \cdot 10^5$ cm. sec^{-1} (for paper), and assuming the velocity of sound in air to be $3.4 \cdot 10^4$ cm. sec^{-1}, the value of α corresponding to the main symmetrical mode of the cone is found to be

$$\alpha_{1,0} = 5.54$$

which is just a little in excess of the largest value taken in Table I.

To draw the directional characteristics of the source, the sound pressure P must be evaluated at a large distance making r to approach infinity; if the common factor: $j\omega\rho \cdot \exp[jkr] \cdot r^{-1}$ is dropped, the limit is found to be proportional to the following value

$$\lim_{r\to\infty} |P| \propto \left| \sum_{n=0}^{\infty} (x_n + jy_n) P_n(\cos\vartheta) \right| \tag{34}$$

Giving to α the above values, and making the angle ϑ to vary, the sum [34] has been calculated from $n = 0$ to $n = 4$, and a set of directional curves has

TABLE I

Values of the $x_n + jy_n$, for different α's and for $U = 1/a^2$

$\alpha =$	0.5	1	2.5	5
$x_0 + jy_0 =$	$0.21106 + j0.039894$	$0.13055 + j0.065955$	$-0.015352 + j0.062044$	$-0.0^245455 - j0.023890$
$x_1 + jy_1 =$	$0.0^253092 + j0.10394$	$0.044426 + j0.17145$	$-0.066823 + j0.16333$	$-0.0^290767 - j0.085201$
$x_2 + jy_2 =$	$-0.012375 - j0.0^227191$	$-0.040445 + j0.0^210278$	$-0.13079 + j0.11888$	$-0.0^255871 - j0.10574$
$x_3 + jy_3 =$	$-0.0^418261 - j0.0^265037$	$-0.0^286243 - j0.0^246973$	$-0.023286 - j0.0251417$	$0.0^222146 - j0.049240$
$x_4 + jy_4 =$	$-0.0562851 - j0.0^233278$	$-0.0^436835 - j0.0^210132$	$-0.0^232570 + j0.012355$	$0.013421 - j0.089106$

been traced in the left half of fig. 3, assuming the pressure on the axis to be unity at every frequency. As the frequency increases, the directionality becomes sharper, and several secondary maxima appear.

The power radiated W can be evaluated integrating the square of pressure divided by ρc, on a sphere whose radius is much larger than wavelength

$$W = \frac{2\pi\omega^2\rho a^4 U^2}{c} \sum_{n=0}^{\infty} \frac{|x_n + jy_n|^2}{2n+1} \tag{35}$$

According to experience the amplitude of the alternating force in the moving coil can be supposed to be constant with frequency, and the vibrations of the diaphragm to be mass-controlled, at least until frequency does not reach the main symmetrical mode. Therefore the product ωU, does not vary any more with frequency, and the changes of power radiated given by [35] are due only to the term in brackets.

As Fig. 4 shows (curve C) power falls rather quickly as frequency increases as it happens with a plane diaphragm in an infinite baffle, or with a pulsating mass-controlled sphere. As it is known, it is only the breaking-off of diaphragm in superior modes which allows the higher frequencies to be radiated at all.

5. Comparison with the equivalent spherical source. To evaluate the approximation obtained, the above results must be compared with those given by the first order approximation, that is with the directional characteristics of an equivalent spherical source, having the same volume, the same strength and the same vibration velocity.

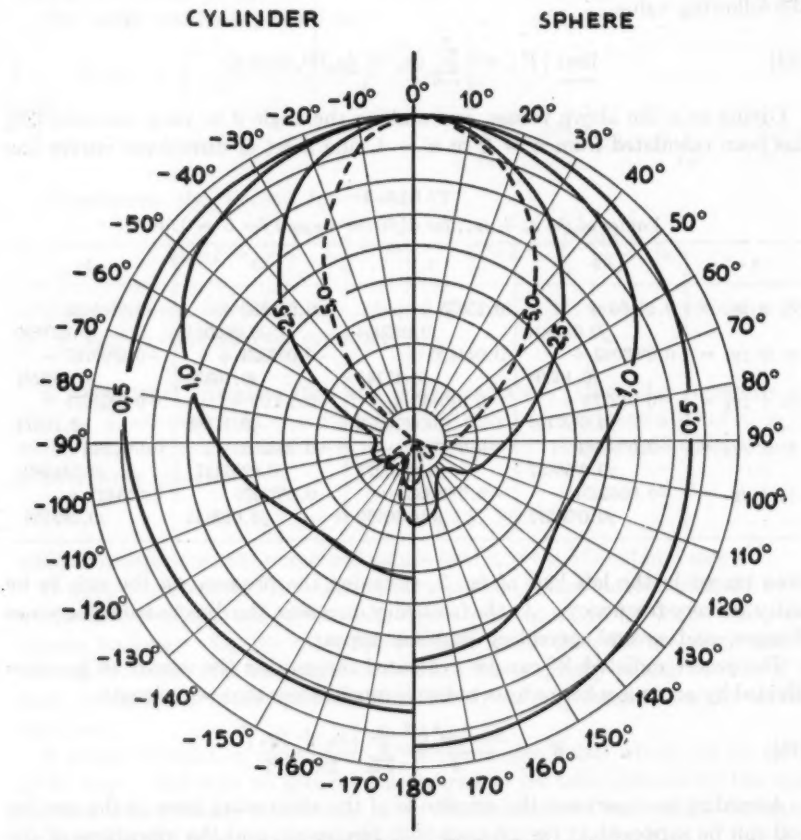

FIG. 3. Directional characteristics of finite cylinder (left) and of equivalent spherical source (right) for $ka = 0.5; 1.0; 2.5; 5.0$.

It is easily found that the diameter of the sphere a_1 must be 1.145 times larger than that of the cylindrical source, and the angle ϑ (fig. 1) is 60° 52′. The new coefficients $x'_n + jy'_n$ are then given (1) (5) by:

$$[36] \qquad x'_n + jy'_n = a_1^2 U \frac{e^{j\alpha}}{F_n(j\alpha)} (n + \tfrac{1}{2}) \int_0^{\vartheta} P_n(\cos\vartheta) \sin\vartheta \cos\vartheta \, d\vartheta$$

while directional characteristics and power can be calculated by the same formulae [34], [35], provided $x'_n + jy'_n$ and a_1 are taken instead of $x_n + jy_n$ and a.

The right half of Fig. 3 gives for the same values of $ka = \alpha$, the new directional characteristics; the agreement is good, specially in the front side, the only difference being that, at high frequency, the sphere, as it could be expected, has a less sharp radiation than the cylinder.

Curve S of Fig. 4 shows that the power radiated by the equivalent sphere

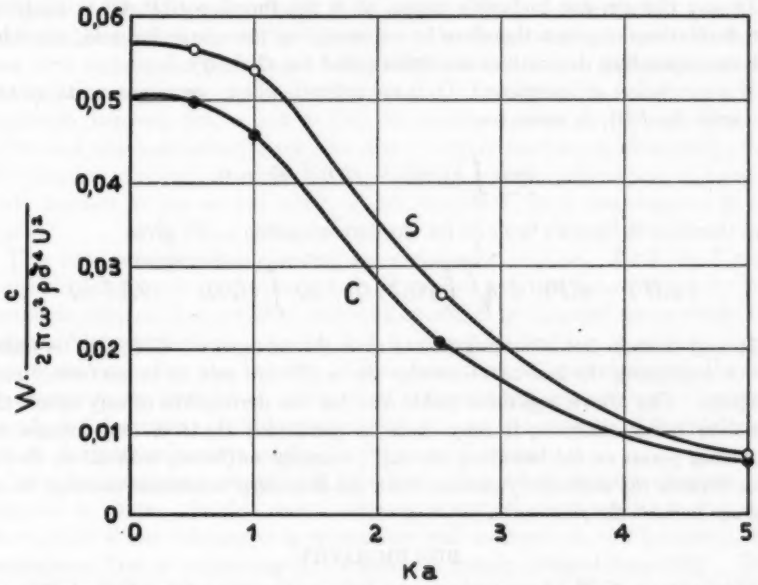

FIG. 4. Variation with ka of power radiated by a mass-controlled diaphragm in cylindrical source (curve C) and in the equivalent spherical source (curve S).

varies with frequency exactly in the same manner as that of the cylinder, being always slightly in excess. This excess does not vanish as frequency approaches zero, the computations for the cylinder being only approximate.

6. Conclusions. The power radiated by a cylindrical sound source with one vibrating basis is calculated together with the directional curves of the same source, evaluating the integral of the wave equation by a minimum method.

The values obtained agree very well with those derived for an equivalent spherical source, and therefore show the soundness of the physical equivalence criterion employed in previous works.

The same agreement proves that further approximations are useless, and that the characteristics calculated coincide accurately enough for all technical purposes, with those of a back-enclosed loudspeaker.

APPENDIX: On the uniform convergence of the approximate solutions. The solution of a partial differential equation of elliptic type can be expressed, under wide conditions, in terms of the boundary conditions referring to the unknown function, or to its normal derivative, by means of the formula:

$$u(P) = \int_S G(P, Q)\, f(Q)\, dQ$$

where $f(Q)$ gives the conditions on the contour S.

In any domain not including points of S the function $G(P, Q)$ is analytic; the derivatives of μ can therefore be expressed by the above formula, provided the corresponding derivatives are substituted for $G(P, Q)$.

If a succession of functions $f_n(Q)$ is considered, which converges *on an average* towards the $f(Q)$, it must be:

$$\lim_{n\to\infty} \int_S |f_n(Q) - f(Q)|^2\, dQ = 0$$

and therefore Schwarz's formula for the corresponding $\mu_n(P)$ gives

$$|u_n(P) - u(P)| \leq \sqrt{\int_S |G(P, Q)|^2\, dQ \cdot \int_S |f_n(Q) - f(Q)|^2\, dQ}$$

In any domain not including points of S the integral of $|G(P, Q)|^2$ is finite; hence combining the last two formulas the $u_n(P)$ are seen to be uniformly convergent. The above argument holds also for the derivatives of any order, the function being analytic; it may then be concluded that: *in any domain not including points on the boundary the $u_n(P)$ converge uniformly with all its derivatives towards the unknown function, when the boundary conditions converge on an average towards the given ones.*

BIBLIOGRAPHY

1. MacLachlan, N. W.: Loudspeakers. 1st Edition, Clarendon Press, Oxford, 1934.
2. Hirayama, A.: Sound radiation from a spherical vibrator. Nippon Electrical Communication Engineering, **8**, 536 (1937).
3. Stenzel, H.: Leitfaden zur Berechung von Schallvorgaengen. J. Springer, Berlin, 1939.
4. Karnovskiy, M.: The directjonal properties of elliptical and spherical sound radiators. Journ. Techn. Phys. U.S.S.R., **10**, 839 and 970 (1940).
5. Bordoni, P. G.: Caratteristiche di radiazione di una sfera rigida munita di calotta oscillante. Commentationes Pontificiae Academiae Scientiarum, **8**, 505 (1944).
6. —— Metodi approssimati per lo studio delle sorgenti sonore. Acta Pontificiae Academiae Scientiarum, **8**, 62 (1944).
7. —— Sulle funzioni di Stokes. Commentationes Pontificiae Academiae Scientiarum **9**, 87 (1945).

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ON A QUESTION OF HADAMARD CONCERNING SUPER-BIHARMONIC FUNCTIONS*

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1. Introduction. If a perpendicular force is applied at some point of a thin, flat, elastic plate which is rigidly clamped on its boundary, will the displacement of the plate be of one sign at all points? This question was raised by Hadamard[1] in his prize memoir on the elastic plate. (The French Academy had posed the prize problem—to integrate the equation of the clamped plate, especially in the case of a rectangular boundary.) The displacement w of a thin plate satisfies the equation $\Delta^2 w(x, y) = \rho(x, y)$ where ρ is the normal pressure and the clamping condition demands that w and its first derivatives vanish on the boundary. If a function which satisfies the relation $\Delta^2 w \geq 0$ be termed super-biharmonic, then the question can be phrased thus: If a function is super-biharmonic in a region and vanishes to the second order on the boundary, is it non-negative in the region?

This note considers this question for rectangular regions. First the Fourier transform method is applied to obtain an explicit solution for an infinitely long rectangle subject to a pressure distribution which is constant transversely but arbitrary lengthwise. It is found that if a positive pressure is applied over a length not exceeding the breadth of the plate, then at large distances the displacement is asymptotic to an exponentially damped sine wave. The wave length and decrement of this wave depend only on the breadth of the plate and not on the particular form of the distribution of pressure.

Next the question is considered for a finite strip which is approximately rectangular in shape. In this case a solution is not obtained explicitly, but it is shown that if the solution w is reasonably well behaved on the boundary, the assumption that w is non-negative violates a certain integral inequality. This integral inequality is of some interest in its own right, and it is conjectured that it is true under very weak restrictions.

The finite region considered is so nearly rectangular that physical intuition would lead one to believe that an approximating rectangle would have the same qualitative behavior. It is to be expected, therefore, that positive and negative deflections would result if a force is applied to the center of a rectangular plate which is about four times as long as it is wide. On the other hand, it is conjectured that for a square plate the answer to Hadamard's question is yes. This conjecture gains some support from the fact that the answer is yes for the circular plate.[2]

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[1] J. Hadamard, "Sur un problème d'analyse relatif à l'equilibre des plaques élastiques encastrées," Memoires présentés par divers savants étrangers à l'Académie des Sciences, 33, n° 4 (1908).

——, "Sur certains cas intéressants au problème biharmonique," *Proceedings of the Fourth International Mathematical Congress at Rome*, 2, pp. 12-14 (1908).

[2] J. H. Michell, "Flexure of circular plates," *Proc. Math. Soc. London*, 34, pp. 223-228 (1901).

2. The infinite strip. To find formal solutions of the equation

$$\partial^4 w/\partial x^4 + 2\partial^4 w/\partial x^2 \partial y^2 + \partial^4 w/\partial y^4 = \rho(x, y) \tag{1}$$

in the infinite strip $-1 \leq y \leq 1$, $-\infty < x < \infty$, it is natural to express w as a Fourier integral.[3] Thus let

$$w(x, y) = \int_{-\infty}^{\infty} e^{ipx}\, u(p, y)\, dp \tag{2}$$

$$\rho(x, y) = \int_{-\infty}^{\infty} e^{ipx}\, g(p, y)\, dp. \tag{3}$$

Substituting (2) and (3) in (1) gives

$$p^4 u - 2p^2\, d^2u/dy^2 + d^4u/dy^4 = g. \tag{4}$$

If the solution w is to represent the displacement of a clamped plate under a distributed load ρ, then w must satisfy the boundary conditions $w = \partial w/\partial y = 0$ for $y = \pm 1$. Therefore a particular solution u of the ordinary differential equation (4) is sought which will give w the correct boundary conditions. It is sufficient for our purpose to have ρ a function of x only; in this case a particular solution is $u = g(p)/p^4 + A \cosh py + By \sinh py$, where A and B are to be chosen so that $u = \partial u/\partial y = 0$ at $y = \pm 1$. It is convenient to let $u = gH$; then we shall verify that

$$H(p, y) = \frac{1}{p^4} + \frac{py \sinh p \sinh py - (\sinh p + p \cosh p) \cosh py}{p^4(p + \cosh p \sinh p)}. \tag{5}$$

Ignoring the case $p = 0$, it is easy to see that $H(p, \pm 1) = 0$. Also

$$\frac{\partial H}{\partial y} = \frac{y \sinh p \cosh py - \cosh p \sinh py}{p^2(p + \cosh p \sinh p)} \tag{6}$$

and $\partial H/\partial y = 0$ for $y = \pm 1$. Thus (5) furnishes a formal solution to the problem.

It is not difficult to give a direct proof that the function w defined by (2) actually satisfies the biharmonic equation (1). Moreover, w and its first derivatives may be shown to vanish at $y = \pm 1$ and at $x = \pm \infty$. To simplify this proof it may be assumed that $\rho(x)$ vanishes outside a finite interval of the x axis. This makes $g(p)$ an entire function. In addition it may be assumed that $\rho(x)$ has a continuous nth derivative. Then by partial integration it is seen that $g(p) = 0(1 + |\, p\,|)^{-n}$. It is to be noted that this relation holds also for complex p if $Im(p)$ is bounded. Choosing a sufficiently large n will make the integrals occurring in the proof absolutely convergent.

At first sight it appears that H has a singularity at $p = 0$; however, expansion

[3] The Fourier integral method has also been applied to the clamped strip by A. Ghizzetti in a paper which appeared after the present note was written, "Ricerche analitiche sul problema dell' equilibrio di una piastra indefinita a forma di striscia, incastrata lungo i due lati," *Univ. Roma Ist. Naz. Alta Mat. Rend. Mat. e Appl.*, (5) 6, pp. 145–187 (1947).

of the numerator in a power series shows that H is analytic at $p = 0$. Further analysis reveals that H is uniformly bounded for $|p| \leq \infty$ and $|y| \leq 1$. The assumed properties of $g(p)$ then guarantee that w defined by (2) is a continuous function which vanishes at $y = \pm 1$. That w vanishes at $\pm\infty$ is a consequence of the Riemann-Lebesgue lemma.

To investigate the asymptotic form of w as $x \to +\infty$, it is necessary to know the behavior of H as a function of the complex variable p. The only possible singularities of H are at the zeros of $1 + \cosh p \sinh p/p$. The zeros of this function are symmetric to the real and imaginary axes, so it is sufficient to consider only the first quadrant. Plotting the loci of the real and imaginary parts equated to zero gives the following information: There is one and only one zero for $Im(p)$ in the intervals $(\pi/2, 3\pi/4)$, $(\pi, 5\pi/4)$, etc. There are no zeros outside these intervals. The real and imaginary parts of the zeros increase together. The first zero is at the point $a + ib = 1.12 + i2.10$.

By Cauchy's theorem

$$w(x, y) = f(x, y) + \int_\delta e^{ipz} Hg \, dp.$$

Here δ is a line parallel to the real axis and a distance c above it, and $f(x, y)$ is the contribution from the poles of H passed over. Clearly

$$\left| \int_\delta e^{ipz} Hg \, dp \right| \leq e^{-cz} \int_\delta |H| \, |g| \, |dp| .$$

Thus $w(x, y) = f(x, y) + 0e^{-cz}$.

Suppose the contour δ passes over only the first two poles of H which are at the points $a + ib$ and $-a + ib$. Evaluating the residue at these points gives, if $a + ib = q$,

$$(7) \qquad f(x, y) = Re\left\{ 2\pi i g(q) e^{iqz} \frac{qy \sinh q \sinh qy - (\sinh q + q \cosh q) \cosh qy}{q^4 \cosh^2 q} \right\}$$

Simplifying (7) and substituting in the expression for w gives

$$(8) \qquad w(x, y) = r e^{-bz} \cos(ax + m) + 0e^{-cz}$$

where r and m are functions of y only and $c > b$. Thus if r is not zero, w oscillates between positive and negative values for x sufficiently large.

It is seen from (7) that $f(x, y)$ could vanish identically only if $g(q) = 0$. Let us now assume that $\rho(x)$ vanishes outside the interval $-\pi/2a \leq x \leq \pi/2a$ and is non-negative inside the interval. Thus

$$Re\{g(q)\} = \frac{1}{2\pi} \int_{-\pi/2a}^{\pi/2a} e^{bz} \cos ax \rho(x) \, dx.$$

Since the integrand is non-negative, $g(q)$ can not vanish unless $\rho(x)$ is identically zero. Thus under these hypotheses on $\rho(x)$ the function $w(x, y)$ eventually behaves like an exponentially damped sine wave of wavelength $2\pi/a = 5.6$ and decrement $e^{\pi b/a} = 360$.

3. The finite strip. The asymptotic solution f given by (7) is actually a biharmonic function ($\Delta^2 f = 0$) which vanishes together with its first derivatives for $y = \pm 1$. We may simplify f by making use of the relation $q + \sinh q \cosh q = 0$ and choosing an appropriate multiplying constant. Thus let

$$f(x, y) = Re\,\{e^{iqx}(\cosh qy - y \coth q \sinh qy)\} \tag{9}$$

Then $f(x, 0) = e^{-2.1x} \cos 1.12x$, so f has nodal lines which intersect the x axis at $1.12x = \pi/2 + n\pi$. Substitution of other values of y shows, after a rather long calculation, that the nodal lines are only slightly curved. They are symmetric about the x axis and are bent away from $+\infty$ with a lag of .26 at $y = \pm 1$, so they deviate about six per cent from a mean vertical line.

Let the region R be bounded by the lines $y = \pm 1$ and the nodal lines which intersect the x axis at $1.12\,x = \pm(3\pi/2)$. Thus R is divided into three congruent sections by the nodal lines which intersect the x axis at $1.12x = \pm\pi/2$. In the central section f is positive, and in the others it is negative. Hence on the boundary S of the region, $f = 0$ and $\partial f/\partial n \geq 0$. Here n is the exterior normal to the boundary.

Let ρ be a non-negative function which is positive somewhere in the central section of R and vanishes outside that section. Let w satisfy $\Delta^2 w = \rho$ in R and $w = \partial w/\partial x = \partial w/\partial y = 0$ on S. We now employ Green's theorem twice:

$$\iint_R (f\Delta^2 w - \Delta w \Delta f) = \int_S (f\partial\Delta w/\partial n - \Delta w \partial f/\partial n)$$

$$\iint_R (w\Delta^2 f - \Delta f \Delta w) = \int_S (w\partial\Delta f/\partial n - \Delta f \partial w/\partial n).$$

Subtracting these relations and making use of the assumed properties of w and f gives

$$\iint_R f\rho = -\int_S \Delta w \partial f/\partial n. \tag{10}$$

Let us assume that $w \geq 0$; then at any point of the boundary where the second derivatives of w are continuous, we have $\Delta w \geq 0$. This follows because if a rotated system of coordinates is chosen with the x axis in the direction of the normal, $\Delta w = \partial^2 w/\partial x^2$. The term $\partial^2 w/\partial y^2$ vanishes because $\partial w/\partial y = 0$ on the boundary and the boundary parallels the y axis at that point. So by the mean value theorem if (x_1, y) is a point of the boundary and (x_2, y) is an interior point,

$$w(x_2, y) = \tfrac{1}{2}(x_1 - x_2)^2 \left(\frac{\partial^2 w}{\partial x^2}\right)_{x = x_3}. \tag{11}$$

Here x_3 lies between x_1 and x_2. Thus $(\partial^2 w/\partial x^2)_{x=x_3} \geq 0$. Allowing x_2 to approach x_1 shows that $(\partial^2 w/\partial x^2)_{x=x_1} \geq 0$.

The integral on the right side of (11) is non-negative, so $\iint_R f\rho \leq 0$. But

this is a contradiction, as f is positive inside the central section and ρ is positive somewhere inside the central section and vanishes outside, which means that $\iint_R f\rho > 0$. The conclusion is that a positive pressure applied over the central section of a clamped plate of the shape R produces a negative deflection at some points.

The relation (10) is, in mechanical language, "a principle of reciprocity"which is demanded by the conservation of energy. Therefore, from a physical point of view, its employment here seems perfectly innocent. But from the abstract viewpoint it has been tacitly assumed that Green's theorem is applicable to the function w. Thus it would appear necessary to assume that the third derivatives of w are bounded in R. To give a proof which involves no assumption on the second or higher derivatives of w, the writer has attempted a more direct approach. It is sufficient to demonstrate the following inequality

$$\iint_R (u\Delta^2 v - v\Delta^2 u)\, dx\, dy \geq 0 \tag{12}$$

when it is given that $u = v = \partial v/\partial n = 0$ and $\partial u/\partial n \geq 0$ on the boundary and $v \geq 0$ inside. A proof of this inequality has been obtained, provided the boundary is sufficiently smooth and has no corners. The fact that R has corners prevents its application here.

An inequality such as (12) might be called a Green's inequality because it is related to Green's theorem. Corresponding to the various forms of Green's theorem we would have similar inequalities. Thus the following theorem (which is the simplest case) is conjectured.

Theorem: *Let f be a continuous non-negative function which vanishes on the boundary of a region R. Let the second derivatives of f be continuous at interior points of R, and let Δf be absolutely integrable over R. Then* $\iint_R \Delta f \leq 0$. The proof of this theorem is easy if R is a circle or sphere.

Clearly the expression (9) defines a biharmonic function vanishing to the second order on the boundary of the strip when q is any root of the equation $1 + \cosh q \sinh q/q = 0$. The set of functions (9) together with the corresponding imaginary parts might be termed the *even separable modes* of the clamped strip. Likewise, the *odd separable modes* are obtained from the real and imaginary parts of $e^{iqz}(\sinh qy - y \tanh q \cosh qy)$ where q satisfies the equation $q - \cosh q \sinh q = 0$. The non-zero roots of $q \pm \cosh q \sinh q = 0$ in the first quadrant can be ordered so that the real and imaginary parts increase together, with the $+$ roots and the $-$ roots alternating. This series begins with a $+$ root, so the first even mode is the least damped of all. If $f_n(x, y)$ represents the sequence of the various modes, it would appear useful in problems concerning clamped strips to seek a solution as a linear combination $\Sigma c_n f_n(x, y)$.

It is to be noted that the simplicity of the expression (9) is deceptive. For example, it is not easy to see by inspection the nature of the nodal lines for the

higher modes; actual calculation shows that they are quite curious.[4] Thus while the wave length of the lowest even mode is longer than that of the next higher even mode, nevertheless the overall length of the positive and negative sections is considerably greater for the higher mode because of a large phase change across the strip. Hence the positive and negative sections of the higher mode are far from being rectangular in shape.

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[4] The writer is indebted to Mr. M. S. Klamkin for carrying out these calculations.

A TABLE OF THE COEFFICIENTS OF THE HERMITE QUADRATURE FORMULA*

ZDENĚK KOPAL

A Gauss-type quadrature formula for an approximate evaluation of definite integrals with doubly infinite limits appears to have first been established by Gourier,[1] who proved that if $f(x)$ is a function of degree not in excess of $2n$-1,

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx = 2^{n+1}\, n!\, \sqrt{\pi} \sum_{i=1}^{n} \frac{f(x_i)}{\{H'_n(x_i)\}^2}, \tag{1}$$

where $H_n(x)$ denotes the Hermite polynomial defined by

$$H_n(x) = e^{x^2} d^n(e^{-x^2})/dx^n \tag{2}$$

and the x_i's in (1) are roots of the Hermite polynomials of n-th order. Numerical values of the Christoffel numbers

$$p_i = 2^{n+1}\, n!/\{H'_n(x_i)\}^2 \tag{3}$$

for $n = 2, 3$, and 4 were given by Berger,[2] and a more complete set of 7D values corresponding to $n = 2(1)9$ was later completed by Reiz;[3] the latter's paper was the first one in which the respective Christoffel numbers were given in decimal form.

In certain computations performed recently in the Center of Analysis of the Massachusetts Institute of Technology, a need arose for the Christoffel numbers of the quadrature formula (1) corresponding to $n = 10(1)20$. Since their values do not appear to have been evaluated before and are apt to be frequently needed in the future, the table which was prepared for the purpose of our investigation will be reproduced below.

In computing the Christoffel numbers as defined by equation (3), use has been made of values of roots of the Hermite polynomials published previously

* The computations involved in the preparation of the foregoing coefficients have been sponsored by the Electronic Research Laboratories, Air Material Command, Cambridge, Massachusetts. The permission of the Commanding Officer of the Laboratories to publish the above data is gratefully acknowledged.

[1] C. R. Acad. Sci., Paris, **97**, 79, 1883.

[2] Nova Acta Regiae Soc. Sci., Uppsala, (3), **16**, No. 4, 1893.

[3] Arkiv för Math., Astr. och Physik, **29** A, No. 29, 1943.

TABLE I

Roots of Hermite polynomials of order ten to twenty and corresponding Christoffel numbers

$n = 10$		$n = 11$		$n = 12$		$n = 13$	
$\pm x$	p	$\pm x$	p	$\pm x$	p	$\pm x$	p
0.342901	0.344642	0	0.369408	0.314240	0.321664	0	0.340092
1.036611	0.135484	0.656810	0.242240	0.947788	0.146967	0.605764	0.237871
1.756684	0.019111 6	1.326557	0.066139	1.597682	0.029117	1.220055	0.079169
2.532732	0.000758 071	2.025948	0.006720 7	2.279507	0.002203 4	1.853106	0.011771
3.436159	0.000004 31065	2.783290	0.000195 62	3.020637	0.000048 372	2.519735	0.000681 24
		3.668471	0.000000 81218	3.889725	0.000000 14099	3.246608	0.000011 527
						4.101337	0.000000 027226

$n = 14$		$n = 15$		$n = 16$		$n = 17$	
$\pm x$	p	$\pm x$	p	$\pm x$	p	$\pm x$	p
0.291745	0.302635	0	0.318260	0.273481	0.286568	0	0.299538
0.878714	0.154083	0.565069	0.23246	0.822951	0.15834	0.531633	0.22671
1.476683	0.038650	1.136115	0.089418	1.380258	0.047285	1.067649	0.097406
2.095184	0.004428 9	1.719993	0.017366	1.951788	0.007266 9	1.612925	0.023087
2.748471	0.000200 3	2.325732	0.001567	2.546202	0.000526 0	2.173503	0.002859
3.462657	0.000002 661	2.967167	0.000056 42	3.176999	0.000015 30	2.757763	0.000168 5
4.304448	0.000000 004868	3.669951	0.000000 5975	3.869448	0.000000 1310	3.378932	0.000004 013
		4.499991	0.000000 000859 0	4.688739	0.000000 000149 8	4.061947	0.000000 02808
						4.871345	0.000000 000025 84

$n = 18$		$n = 19$		$n = 20$	
$\pm x$	p	$\pm x$	p	$\pm x$	p
0.258268	0.272783	0	0.283773	0.245341	0.260793
0.776683	0.16068	0.503520	0.22094	0.737473	0.16174
1.300921	0.054895	1.010369	0.10360	1.234075	0.061508
1.835532	0.010516	1.524171	0.028667	1.738538	0.01400
2.386299	0.001066	2.049232	0.004507	2.254974	0.001830
2.961377	0.000051 80	2.591134	0.000378 5	2.788806	0.000128 8
3.573769	0.000001 019	3.157849	0.000015 35	3.347854	0.000004 40
4.248118	0.000000 005906	3.762187	0.000000 253	3.944764	0.000000 0612
5.048364	0.000000 000004 417	4.428533	0.000000 00122	4.603682	0.000000 000248
		5.220271	0.000000 000000 748	5.387481	0.000000 000000 126

by Smith;[4] their accuracy of six decimal figures imposed the limit to the accuracy with which the corresponding Christoffel numbers could be evaluated. As many decimals are retained in our table as are regarded to be significant; the error of no published value of p is expected to exceed one unit of the last place.

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[4] Amer. Math. Monthly, **48**, 354, 1936. Smith defined his polynomials as

$$U_n(x) = (-1)^n e^{x^2/2} d^n(e^{-x^2/2})/dx^n$$

which are related with the Hermite polynomials as defined by equation (2) above by

$$H_n(x) = (-\sqrt{2})^n U_n(x\sqrt{2}).$$

In consequence, the roots of the U-polynomials as evaluated by Smith must be divided by $\sqrt{2}$ to obtain the requisite roots of $H_n(x)$. The roots so divided will be given in our table together with the corresponding Christoffel numbers.

GENERALIZATION OF APPELL'S TRANSFORMATION*

By Edward Kasner and John De Cicco

1. Introduction. We shall extend Appell's Theorem concerning the transformation theory of the dynamical trajectories of a field of force from positional fields to generalized fields where the force vector depends not only on the position of the point but also on the direction through the point. Positional fields may be described as *isotropic* and generalized fields as *anisotropic*.[1]

In 1889, Appell showed that the appropriate transformations for positional fields are those of the eight-parameter collineation group together with a time factor which depends only on the particular collineation used up to an arbitrary constant.[2] We shall prove that *the required transformations for generalized fields are those of the complete mixed projective group of eight-parameters consisting of collineations and correlations together with a time factor.* This time factor depends in the case of a collineation only on the particular collineation up to an arbitrary function of two variables, and in the case of a correlation, it depends not only on the particular correlation, but also on the particular generalized field under consideration up to an arbitrary function of two variables.

2. The transformation theory of generalized fields of force. If (ϕ, ψ) are the rectangular components of a generalized field of force in the plane, the corresponding equations of motion of a particle of unit mass are

$$d^2x/dt^2 = \phi(x, y, p),\ d^2y/dt^2 = \psi(x, y, p). \tag{1}$$

The cartesian coordinates of a point are denoted by (x, y), the slope through the point by $p = dy/dx$, and the time by t. If the force vector is not identically zero, there are ∞^3 trajectories in a generalized field of force.

Before proceeding, it is remarked that

$$\frac{dp}{dt} = \frac{d}{dt}\left(\frac{dy/dt}{dx/dt}\right) = \frac{\psi - p\phi}{dx/dt}. \tag{2}$$

To the differential equation (1), we apply a general contact transformation

$$X = X(x, y, p),\qquad Y = Y(x, y, p),$$
$$P = P(x, y, p) = \frac{Y_x + pY_y}{X_x + pX_y} = \frac{Y_p}{X_p}, \tag{3}$$

where the jacobian $J = \partial(X, Y, P)/\partial(x, y, p)$ is not zero in a certain region of lineal-elements (x, y, p), together with a change of time of the form

$$dT = dt/F(x, y, p). \tag{4}$$

This type of transformation defined by (3) and (4) depends not only on the union: $x = x(t)$, $y = y(t)$, $p = p(t) = dy/dx$, under consideration but also on the particular form of the parameterization of the union.

* Presented to the American Mathematical Society, 1948.

It is seen that the components of the velocity vectors of the trajectories of (1) undergo by use of (2), the following change

$$(5)\qquad \begin{aligned} \frac{dX}{dT} &= F\left[(X_x + pX_y)\frac{dx}{dt} + \frac{(\psi - p\phi)}{dx/dt}X_p\right], \\ \frac{dY}{dT} &= F\left[(Y_x + pY_y)\frac{dx}{dt} + \frac{(\psi - p\phi)}{dx/dt}Y_p\right]. \end{aligned}$$

Thus by use of (2), the differential equations (1) under the transformation defined by (3) and (4), assume the form

$$(6)\qquad \begin{aligned} \frac{d^2X}{dT^2} &= F^2\phi(X_x + pX_y) + F(\psi - p\phi)\frac{\partial}{\partial p}[F(X_x + pX_y)] \\ &\quad + F\left(\frac{\partial}{\partial x} + p\frac{\partial}{\partial y}\right)[F(\psi - p\phi)X_p] + F\left(\frac{dx}{dt}\right)^2\left(\frac{\partial}{\partial x} + p\frac{\partial}{\partial y}\right)[F(X_x + pX_y)] \\ &\quad + \frac{F(\psi - p\phi)}{(dx/dt)^2}\left\{-F\phi X_p + \frac{\partial}{\partial p}[F(\psi - p\phi)X_p]\right\}, \\ \frac{d^2Y}{dT^2} &= F^2\phi(Y_x + pY_y) + F(\psi - p\phi)\frac{\partial}{\partial p}[F(Y_x + pY_y)] \\ &\quad + F\left(\frac{\partial}{\partial x} + p\frac{\partial}{\partial y}\right)[F(\psi - p\phi)Y_p] + F\left(\frac{dx}{dt}\right)^2\left(\frac{\partial}{\partial x} + p\frac{\partial}{\partial y}\right)[F(Y_x + pY_y)] \\ &\quad + \frac{F(\psi - p\phi)}{(dx/dt)^2}\left\{-F\phi Y_p + \frac{\partial}{\partial p}[F(\psi - p\phi)Y_p]\right\}. \end{aligned}$$

Thus under a general transformation of the form defined by (3) and (4), the dynamical trajectories of a generalized field of force are *not* in general converted into the dynamical trajectories of a generalized field of force.

By (6), it is found that *the transformed trajectories are those of a generalized field of force if and only if the transformation defined by* (3) *and* (4), *obeys the four conditions*

$$(7.1)\qquad F(X_{xx} + 2pX_{xy} + p^2X_{yy}) + (F_x + pF_y)(X_x + pX_y) = 0,$$

$$(7.2)\qquad F(Y_{xx} + 2pY_{xy} + p^2Y_{yy}) + (F_x + pF_y)(Y_x + pY_y) = 0,$$

$$(7.3)\qquad (\psi - p\phi)(FX_{pp} + F_pX_p) + FX_p(\psi_p - p\phi_p - 2\phi) = 0,$$

$$(7.4)\qquad (\psi - p\phi)(FY_{pp} + F_pY_p) + FY_p(\psi_p - p\phi_p - 2\phi) = 0.$$

We shall proceed to discuss this system of equations (7).

3. Transformations converting every generalized dynamical family of trajectories into a generalized dynamical family. From the equations (7), we shall deduce the following result.

Theorem 1. A transformation of the form (3) *and* (4), *carries the* ∞^3 *trajectories of a generalized field of force into the* ∞^3 *trajectories of a generalized field of force*

if and only if the contact transformation (3) *is an extended collineation*

$$X = \frac{a_1 x + b_1 y + c_1}{a_0 x + b_0 y + c_0}, \qquad Y = \frac{a_2 x + b_2 y + c_2}{a_0 x + b_0 y + c_0},$$

(8)

$$P = \frac{C_1(y - xp) + B_1 + A_1 p}{C_2(y - xp) + B_2 + A_2 p},$$

where the capital letters denote minors in the determinant $| a_1 b_2 c_0 | \neq 0$, *of the nine constants, and the change of time* (4) *is of the form*

$$dT = \frac{dt}{k(u, v)(a_0 x + b_0 y + c_0)^2}, \tag{9}$$

where k *is an arbitrary function of the two variables* $u = p$ *and* $v = -y + px$ *only.*

Moreover if the equations of motion of a particle in the transformed generalized field of force are

$$d^2X/dT^2 = \Phi(X, Y, P), \qquad d^2Y/dT^2 = \Psi(X, Y, P), \tag{10}$$

the relation between the new field and the original field is explicitly as follows

$$\begin{aligned}\Phi &= k^2(a_0 x + b_0 y + c_0)^2[C_2(y\phi - x\psi) + B_2\phi + A_2\psi] \\ &\quad + k(a_0 x + b_0 y + c_0)^2[C_2(y - xp) + B_2 + A_2 p](\psi - p\phi)(k_u + xk_v),\end{aligned} \tag{11}$$

$$\begin{aligned}\Psi &= k^2(a_0 x + b_0 y + c_0)^2[C_1(y\phi - x\psi) + B_1\phi + A_1\psi] \\ &\quad + k(a_0 x + b_0 y + c_0)^2[C_1(y - xp) + B_1 + A_1 p](\psi - p\phi)(k_u + xk_v).\end{aligned}$$

If it be demanded that both the original field and the transformed field be positional, it can be shown by (11) that k is constant. In this case, our Theorem 1 reduces to Appell's Theorem.

To prove Theorem 1, we first note that the equations (7.3) and (7.4) must be identities in ϕ, ψ, ϕ_p, ψ_p. Upon placing the coefficients equal to zero, we find that $X_p = 0$ and $Y_p = 0$. Thus the transformation (3) is necessarily a point transformation.

Next eliminating F from the equations (7.1) and (7.2), we have

$$\begin{aligned}&(X_x + pX_y)(Y_{xx} + 2pY_{xy} + p^2Y_{yy}) \\ &\qquad - (Y_x + pY_y)(X_{xx} + 2pX_{xy} + p^2X_{yy}) = 0.\end{aligned} \tag{12}$$

But this is the condition that the point-to-point transformation (3) carry every straight line into a straight line. Thus (3) is the extended collineation (8).

From either one of the equations (7.1) or (7.2) together with the equation (8), we obtain (9). From (8) and (9), the equations (11) follow. This completes the proof of Theorem 1.

By the transformation (8) and (9), the trajectories of the original field are converted into the trajectories of the new field. Also the directions of the

forces of the two fields are projectively related. However the two fields of force are *not* projectively related. The relationship between the corresponding force vectors of the two fields is quite complicated. The positional case has been discussed by Kasner.[3]

4. Transformations sending only one generalized dynamical family of trajectories into a dynamical family. Henceforth we can assume that either $X_p \neq 0$ or $Y_p \neq 0$ or both $X_p \neq 0$ and $Y_p \neq 0$. We shall establish the following result.

Theorem 2. A contact transformation (3), *which is not an extended collineation, will send the* ∞^3 *dynamical trajectories of a generalized field of force into the* ∞^3 *dynamical trajectories of a generalized field of force if and only if it is an extended correlation*

$$X = \frac{a_1u + b_1v + c_1}{a_0u + b_0v + c_0}, \qquad Y = \frac{a_2u + b_2v + c_2}{a_0u + b_0v + c_0},$$

$$P = \frac{C_1(v - uw) + B_1 + A_1w}{C_2(v - uw) + B_2 + A_2w} \tag{13}$$

where the capital letters denote the minors in the determinant $|a_1b_2c_0| \neq 0$, *of the nine constants, and* (u, v, w) *are given by the Legendre transformation*

$$u = p, \qquad b = -y + xp, \qquad w = x, \tag{14}$$

which is polarity with respect to the parabola $y^2 = 2x$. *For the given generalized field of force with rectangular components* (ϕ, ψ), *the appropriate change in the time* (4) *is*

$$dT = \frac{(\psi - p\phi)\exp\left(-\int[\phi/(\psi - p\phi)]\,dp\right)dt}{k(x, y)(a_0u + b_0v + c_0)^2}, \tag{15}$$

where k *is an arbitrary function of two variables* x *and* y.

Moreover the relation between the new field and the original field is as follows

$$\Phi = \frac{k}{\psi - p\phi}\exp\left(2\int[\phi/(\psi - p\phi)]\,dp\right)$$

$$\left[(-C_2y + B_2 + A_2x)\left(k_x + pk_y + k\left\{\frac{\partial}{\partial x} + p\frac{\partial}{\partial y}\right\}\int\frac{\phi}{\psi - p\phi}\,dp\right) + k(-C_2p + A_2)\right], \tag{16}$$

$$\Psi = \frac{k}{\psi - p\phi}\exp\left(2\int[\phi/(\psi - p\phi)]\,dp\right)$$

$$\left[(-C_1y + B_1 + A_1x)\left(k_x + pk_y + k\left\{\frac{\partial}{\partial x} + p\frac{\partial}{\partial y}\right\}\int\frac{\phi}{\psi - p\phi}\,dp\right) + k(-C_1p + A_1).\right]$$

If the original field is positional, it is found by these formulas (16) that the new field can never be positional except in certain very special cases. This verifies Appell's Theorem that in the positional case, the only possible transformations are the collineations with the appropriate changes in the time.

By the transformation (13) and (15), the trajectories of the original field are carried into the trajectories of the new field. However the directions of the forces are *not* related by the correlation (13), and hence the two fields of force are *not* related by the correlation. It is noticed that the change of the time (15) depends essentially on the original field. *This is a new phenomenon.*

To prove our Theorem 2, we proceed as follows. First define λ by the relation

$$\lambda = (\psi - p\phi)\exp\Big(-\int[\phi/(\psi - p\phi)]\,dp\Big). \tag{17}$$

By this and (14), it is found that the equations (7) may be written in the form

$$\begin{gathered} FX_{ww} + F_wX_w = 0, \qquad FY_{ww} + F_wY_w = 0, \\ X_{pp}/X_p = Y_{pp}/Y_p = -(F_p/F) - (\lambda_p/\lambda). \end{gathered} \tag{18}$$

Eliminating F and λ from the equations (18), we obtain the conditions

$$X_wY_{ww} - Y_wX_{ww} = 0, \qquad X_pY_{pp} - Y_pX_{pp} = 0. \tag{19}$$

Since either $X_p \neq 0$, or $Y_p \neq 0$, or both $X_p \neq 0$ and $Y_p \neq 0$, the last condition states that any point (x, y) of the (x, y)-plane is converted into a line (U, V) where $U = P$ and $V = -Y + XP$, of the (X, Y)-plane. On the other hand, if either $X_w \neq 0$, or $Y_w \neq 0$, or both $X_w \neq 0$ and $Y_w \neq 0$, the first equation states that every line (u, v) where $u = p$ and $v = -y + xp$, of the (x, y)-plane corresponds to a line (U, V) of the (X, Y)-plane. Since the contact transformation is assumed to be one-to-one, this is impossible. Hence it follows that $X_w = 0$ and $Y_w = 0$. Therefore our contact transformation is point-to-line and line-to-point, and hence is an extended correlation (13).

That the change in the time is of the form (15), is now an immediate consequence of (13), (17), and the last of equations (18). This completes the proof of Theorem 2.

5. Differential equations of the type $(G)^4$. Upon eliminating the time t from the differential equations (1) of motion of a particle in a generalized field of force, we find that *the differential equation of third order defining a generalized dynamical family, is*

$$(\psi - p\phi)\frac{d^3y}{dx^3} = [\psi_x + p(\psi_y - \phi_x) - p^2\phi_y]\frac{d^2y}{dx^2} + [\psi_p - p\phi_p - 3\phi]\Big(\frac{d^2y}{dx^2}\Big)^2. \tag{20}$$

Evidently this differential equation is *of the type* (G)

$$\frac{d^3y}{dx^3} = G(x, y, p)\,\frac{d^2y}{dx^2} + H(x, y, p)\,\Big(\frac{d^2y}{dx^2}\Big)^2. \tag{21}$$

Conversely we have proved elsewhere that *any differential equation of the type (G) always represents the* ∞^3 *dynamical trajectories of a generalized field of force.* However in the positional case, the functions G and H must be specialized further. Kasner obtained *a completely characteristic set of five geometric properties* for the ∞^3 dynamical trajectories of a positional field of force.

Theorem 3. An arbitrary lineal-element transformation, not necessarily of the contact type, will send the ∞^3 *dynamical trajectories of every generalized field of force into the* ∞^3 *dynamical trajectories of a generalized field of force if and only if it is an extended collineation or correlation.*

A general theorem of Kasner states that *any non-contact transformation of lineal-elements carries exactly* ∞^2 *unions into unions*[5]. Hence our lineal-element transformation must be a contact transformation since it carries at least ∞^3 unions into unions. By a previously stated theorem, every generalized dynamical family is identical with the type (G). Our Theorem 3 then follows from another theorem of Kasner which states that *all the contact transformations converting every system of* ∞^3 *curves of the type* (G) *into a system of* ∞^2 *curves of the type* (G) *are the extended collineations and correlations.*

It is remarked that Theorem 3 differs essentially from the Theorems 1 and 2 which are generalizations of Appell's Transformations. For in Theorems 1 and 2, we are concerned with the parametric forms of the trajectories, namely, $x = x(t)$, $y = y(t)$, $p = p(t) = dy/dx$, and consider transformations operating on the *four* variables $(x, y, p; t)$; whereas in Theorem 3, we are dealing with the nonparametric form of the trajectories, namely, $y = y(x)$, $p = p(x) = dy/dx$, and consider lineal-element transformations, that is, those operating on only *three* variables (x, y, p).

6. The system S_k. A system S_k of curves in a generalized field of force consists of curves along which a constrained motion is possible so that the pressure P is proportional to the normal component of the force vector N. Thus $P = kN$.

The differential equation of third order defining any system S_k *is*

$$(\psi - p\phi)\frac{d^3y}{dx^3} = [\psi_x + p(\psi_y - \phi_x) - p^2\phi_y]\frac{d^2y}{dx^2} + [\psi_p - p\phi_p - 3\phi - \frac{(n-2)(\phi + p\psi)}{1+p^2}]\left(\frac{d^2y}{dx^2}\right)^2. \tag{22}$$

There are ∞^3 *curves in any system* S_k *for a fixed* k. *The relation between* n *and* k *is*

$$n = 2/(k+1) \tag{23}$$

It is noted that $n = 2$ or $k = 0$ gives the system S_0 of generalized dynamical trajectories, whereas $n = 0$ or $k = \infty$ gives the system S_∞ of velocity curves. A curve is a *velocity curve* corresponding to the speed v_0 if any trajectory is started from a lineal-element of the curve with that speed is initially osculated by the curve. The case $k = 1$ or $n = 1$ gives the system S_1 of *general catenaries.* These are the curves formed from the resulting equilibrium when an inextensible flexible

homogeneous string is suspended in any generalized field of force. We shall define the case $k = -2$ or $n = -2$ as the system S_{-2} of *brachistochrones*. These will be true brachistochrones, that is, curves along which the time of the constrained motion between any two points is least, if the field of force is positional and conservative[6].

A curve is a *curvature trajectory* of a family of ∞^2 curves: $d^2y/dx^2 = f(x, y, p)$, if its curvature at any point is a constant multiple of the curvature of the tangent curve of the given family at the given point[7].

The differential equation of third order defining a curvature family is

$$\frac{d^3y}{dx^3} = (\alpha_x + p\alpha_y)\frac{d^2y}{dx^2} + \alpha_p\left(\frac{d^2y}{dx^2}\right)^2, \tag{24}$$

where α is an arbitrary function of three variables (x, y, p). Thus any curvature family consists of ∞^3 curves of the type (G) of special form.

We have proved that *except for the velocity systems S_∞, any family of curves of the type (G) always represents a system S_k in a generalized field of force. On the other hand, a family of curves of the type (G) is a velocity system S_∞ if and only if it is a system of curvature trajectories.*

In the positional case, Kasner obtained a *characteristic set of five geometric properties* for any system S_k.

Theorem 4. An arbitrary lineal-element transformation, not necessarily of the contact type, will send the ∞^3 curves of a system S_k, including the velocity systems $k = \infty$, into the ∞^3 curves of a system S_k if and only if it is an extended collineation or correlation.

For $k \neq \infty$, the proof of this result is similar to the proof of Theorem 3. Since a velocity system S_∞ is of the type (G), it follows that the lineal-element transformation must be either an extended collineation or correlation. That a velocity system is converted under an extended collineation or correlation, into a velocity system follows from a Theorem of Mehmke-Segre which states that *the ratio of the curvatures of two tangent curves is invariant under the collineation group*, and the fact that *the ratio of the curvatures is changed into its reciprocal under any correlation.*

In the positional case, we have proved that *the only transformations which convert every system S_k, except for the system S_0 of trajectories, into a system S_k, are the similitudes.* Of course, the appropriate transformations for the system S_0 of trajectories are the collineations.

REFERENCES

1. Kasner and De Cicco: *A generalized theory of dynamical trajectories*, Trans. Amer. Math. Soc. **54** (1943), pp. 23–38.
2. Appell: Amer. Journ. Math. (1889).
3. Kasner: *Differential-geometric aspects of dynamics*, Amer. Math. Soc. Colloquium Publications, vol. 3, 1913, 1934 (referred to as the Princeton Colloquium). Also see series of papers in Trans. Amer. Math. Soc., vols. 7–11 (1906–1910).
4. See the Princeton Colloquium, pp. 11–12, and pp. 77–78.

5. KASNER: *General transformation theory of differential-elements*, Amer. Journ. Math., **32** (1904), pp. 392–401.
6. See the Princeton Colloquium, pp. 91–97. Also reference 1.
7. KASNER: *Dynamical trajectories and curvature trajectories*, Bull. Amer. Math. Soc., **40** (1934), pp. 449–455.
8. KASNER AND DE CICCO: *Transformation theory of physical curves*, Proc. Nat. Acad. Sci. (1947).

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CALCULATION OF THE DISPLACEMENTS IN PLASTIC TORSION*

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1. Introduction. We consider the problem of torsion of a prismatical bar assuming that the material exhibits strain-hardening and the bar has any cross section. Our purpose is to derive the displacements by making a minimum of assumptions regarding the stress distribution.

The basic assumptions are:

a) The principal axes of stress are always parallel to the principal axes of strain.

b) The figure consisting of the three principal strain circles of Mohr is always geometrically similar to the figure of the three stress circles of Mohr.

c) The incompressibility condition is satisfied.

d) For each material the intensity D_d of the strain deviation is a known function of the intensity of the stress deviation T_d:

$$D_d = F(T_d)$$

in which

$$T_d = [(\sigma_x - \sigma_m)^2 + (\sigma_y - \sigma_m)^2 + (\sigma_z - \sigma_m)^2 + 2(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)]^{\frac{1}{2}},$$

$$D_d = [\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2 + \tfrac{1}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)]^{\frac{1}{2}}$$

$$\sigma_m = (\sigma_x + \sigma_y + \sigma_z)/3$$

By means of these assumptions we get the stress–strain relations

$$\epsilon_x = [F(T_d)/T_d](\sigma_x - \sigma_m), \qquad \gamma_{xy} = 2[F(T_d)/T_d]\tau_{xy}$$

The remaining formulae are obtained by cyclic permutation of the subscripts x, y, z.

Let us now consider the bar twisted by couples applied at the ends. The axis z is assumed to be parallel to the generators of the cylindrical surface of the bar. We put

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$$

Assuming that no body forces are acting on the bar we find from the equations of equilibrium that we can introduce a stress function $\phi(x, y)$ such that

$$\tau_{xz} = \partial\phi/\partial y, \qquad \tau_{yz} = -\partial\phi/\partial z$$

The stress distribution will be known if $\phi(x, y)$ is determined.

From the stress–strain relations we find

$$\epsilon_x = \epsilon_y = \epsilon_z = \gamma_{xy} = 0 \tag{1}$$

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and

$$\gamma_{xz} = \frac{2F(\sqrt{2(\phi_x^2 + \phi_y^2)})}{\sqrt{2(\phi_x^2 + \phi_y^2)}}\,\phi_y, \qquad \gamma_{yz} = -\frac{2F(\sqrt{2(\phi_x^2 + \phi_y^2)})}{\sqrt{2(\phi_x^2 + \phi_y^2)}}\,\phi_x \tag{2}$$

Introducing the Eqs. (1) and (2) into the compatibility equations we find

$$\partial\gamma_{xz}/\partial y - \partial\gamma_{yz}/\partial x = t = \text{const.} \tag{3}$$

and the differential equation for the stress function $\phi(x, y)$:

$$\frac{\partial}{\partial x}\left[\frac{F(\sqrt{2(\phi_x^2 + \phi_y^2)})}{\sqrt{2(\phi_x^2 + \phi_y^2)}}\,\phi_x\right] + \frac{\partial}{\partial y}\left[\frac{F(\sqrt{2(\phi_x^2 + \phi_y^2)})}{\sqrt{2(\phi_x^2 + \phi_y^2)}}\,\phi_y\right] = \frac{t}{2} \tag{4}$$

Eq. (4) has first been given by A. Nadai[2].

The boundary conditions are the same as in the theory of elasticity. We find that the stress function ϕ must be constant along the boundary of the cross section. If the cross section is simply connected then we can choose this constant arbitrarily and we may put $\phi = 0$ at the boundary. We find also

$$M_t = 2\iint \phi\, dx\, dy \tag{5}$$

for the torque. The resultant force on a cross section is zero, exactly as in the elastic theory.

By integrating Eq. (4) we find $\phi = \phi(x, y, t)$ and introducing this function into the equation (5) we can express t as a function of M_t only.

2. Displacements in the torsion of prismatical bars. Suppose the stress function $\phi(x, y)$ has been determined for a given cross section. Then we can find easily the expressions for the displacements. Putting

$$\frac{2F(\sqrt{2(\phi_x^2 + \phi_y^2)})}{\sqrt{2(\phi_x^2 + \phi_y^2)}}\,\phi_y = f_1(x, y), \qquad -\frac{2F(\sqrt{2(\phi_x^2 + \phi_y^2)})}{\sqrt{2(\phi_x^2 + \phi_y^2)}}\,\phi_x = f_2(x, y),$$

and using the equations (1) we find

$$\partial u/\partial x = 0, \qquad \partial v/\partial y = 0, \qquad \partial w/\partial z = 0,$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0, \qquad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = f_1(x, y), \qquad \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = f_2(x, y) \tag{6}$$

where f_1 and f_2 are known functions.

From the third of the equations (6) we find $w = w(x, y)$ and from the fifth and sixth of them we get $\partial u/\partial z = f_1 - w_x$, $\partial v/\partial z = f_2 - w_y$. Integrating we find

$$u = z(f_1 - w_x) + u_0(x, y), \qquad v = z(f_2 - w_y) + v_0(x, y) \tag{7}$$

Substituting from the equations (7) into the first and second on the equations (6) we find

$$z(f_{1x} - w_{xx}) + u_{0x} = 0, \qquad z(f_{2y} - w_{yy}) + v_{0y} = 0 \tag{8}$$

These equations must be valid for every value of z. Hence $u_{0z} = 0$, $v_{0z} = 0$. Therefore $u_0(x, y) = u_0(y)$, $v_0(x, y) = v_0(x)$.

Substituting from the equations (7) into the fourth of the equations (6) we find

$$z(f_{1y} + f_{2x} - 2w_{xy}) + u_0' + v_0' = 0 \tag{9}$$

This equation also must be valid for every z. Therefore $u_0'(y) + v_0'(x) = 0$ and $u_0 = -v_0 = k = \text{const}$. Thus u, v, w have the following values

$$u = z(f_1 - w_x) + k, \qquad v = z(f_2 - w_y) - k, \qquad w = w(x, y) \tag{10}$$

It remains to determine the function $w = w(x, y)$. As the equations (8) and (9) must be valid for every z we get

$$w_{xx} = f_{1x}, \qquad w_{yy} = f_{2y}, \qquad w_{xy} = \tfrac{1}{2}(f_{1y} + f_{2x}) \tag{11}$$

Therefore

$$w_x = f_1 + p_1(y), \qquad w_y = f_2 + p_2(x) \tag{12}$$

where p_1 and p_2 are two unknown functions. From the equations (12) we find

$$w_{xy} = f_{1y} + p_1'(y) = f_{2x} + p_2'(x)$$

and

$$w_{xy} = \tfrac{1}{2}(f_{1y} + f_{2x}) + \tfrac{1}{2}(p_1'(y) + p_2'(x)) \tag{13}$$

Substituting from equation (13) into the third equation (11) we find $p_1'(y) + p_2'(x) = 0$. Hence $p_1'(y) = -p_2'(x) = \lambda = \text{const}$. Therefore $p_1 = \lambda y + \mu$, $p_2 = -\lambda x + \nu$ in which λ, μ, ν are constants. Hence equations (10) become

$$u = -\lambda yz - \mu z + k, \qquad v = \lambda xz - \nu z - k \tag{14}$$

and for the function w we have

$$w_x = f_1 + \lambda y + \mu, \qquad w_y = f_2 - \lambda x + \nu \tag{15}$$

The equations (14) and (15) give the displacements u, v, w.

The four constants λ, μ, ν, k can be determined from the conditions of fixing. The constant t in the equation (4) can be related to these constants. Indeed from the equation (3) we get $f_{1y} - f_{2x} = t$. But from the equations (15) we have $w_{xy} = f_{1y} + \lambda = f_{2x} - \lambda$. Hence $f_{1y} - f_{2x} = -2\lambda$ and $t = -2\lambda$. Assuming $u = v = 0$ for $x = y = 0$ but for any z, we get $k = \mu = \nu = 0$. In this case we have $u = -\lambda yz$, $v = \lambda xz$. These equations represent a rotation of the cross sections about the z-axis with λ representing the angle of twist per unit length of the bar, as in the elastic solution. The relation between t and M_t is of course not the same.

3. Example. We shall show now how we can apply the above formulae for the solution of concrete problems.

Let us assume that the function $D_d = F(T_d)$ has the form

$$D_d = A \cdot T_d^3 \tag{16}$$

where A is a constant. Such a function represents the behavior of aluminum with some accuracy.

With this assumption Eq. (4) becomes

$$\phi_{xx}(3\phi_x^2 + \phi_y^2) + \phi_{yy}(3\phi_y^2 + \phi_x^2) + 4\phi_x\phi_y\phi_{xy} = t/4A \tag{17}$$

We consider the case of a narrow rectangular cross section, the boundaries of which are given by the equations $x = \pm a$, $y = \pm b$. It is assumed that b is very large in comparison with a.

Neglecting the effect of the short sides of the rectangle we put $\phi_y = 0$. Hence Eq. (17) becomes

$$\phi_{xx} \cdot \phi_x^2 = t/12A \tag{18}$$

The solution of this differential equation is given by

$$\phi = (3t^{1/3}/4^{4/3}A^{1/3})(x - C_1)^{4/3} + C_2$$

where C_1 and C_2 are integration constants. By putting

$$C_1 = 0, \qquad C_2 = -(3t^{1/3}/4^{4/3}A^{1/3})a^{4/3}$$

we find

$$\phi = (3t^{1/3}/4^{4/3}A^{1/3})(x^{4/3} - a^{4/3}) \tag{19}$$

with $\phi = 0$ for $x = \pm a$.

We have now

$$f_1(x, y) = 4A(\phi_x^2 + \phi_y^2)\phi_y = 0$$

$$f_2(x, y) = -4A(\phi_x^2 + \phi_y^2)\phi_x = -4A\phi_x^3 = -t_z$$

Using the Eqs. (14) and (15) and remembering that $t = -2\lambda$ we find

$$u = -\lambda yz, \qquad v = \lambda xz, \qquad w_x = \lambda y, \qquad w_y = \lambda x, \tag{20}$$

where we have assumed that $k = \mu = \nu = 0$. Integrating we find

$$w = \lambda xy. \tag{20a}$$

For the calculation of the torque we can use the equation (5).

BIBLIOGRAPHY

1. Prager, W.: An Introduction to the Mathematical Theory of Plasticity. Journal of Applied Physics, **18**, 375 (1947).
2. Nadai, A.: The Creep of Metals Under Various Stress Conditions. Th. v. Kármán Anniversary Volume, 237 (1941).

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COEFFICIENTS FOR FACILITATING TRIGONOMETRIC INTERPOLATION

By Herbert E. Salzer

The trigonometric sum of the n^{th} order, $f(x) = C_0 + (C_1 \cos x + S_1 \sin x) + \cdots + (C_n \cos nx + S_n \sin nx)$, which assumes the values $f_0, f_1, \cdots, f_{2n}$ when $x = x_0, x_1, \cdots, x_{2n}$, i.e. determined by an odd number of points, is expressible in the well-known form

$$(1) \qquad f(x) = \sum_{i=0}^{2n} \frac{\sin \frac{1}{2}(x - x_0) \sin \frac{1}{2}(x - x_1) \cdots \sin \frac{1}{2}(x - x_{2n})}{\sin \frac{1}{2}(x_i - x_0) \sin \frac{1}{2}(x_i - x_1) \cdots \sin \frac{1}{2}(x_i - x_{2n})} f_i,$$

where the coefficient of f_i is of the form $\prod'^{2n}_{j=0} \sin \frac{1}{2}(x - x_j) / \prod'^{2n}_{j=0} \sin \frac{1}{2}(x_i - x_j)$, the symbol Π' denoting the absence of a factor having $j = i$. Formula (1) is commonly known as Gauss's formula for trigonometric interpolation. Even when the points x_i are equally spaced, there is a large amount of computational labor involved in the use of (1). The purpose of this present article is to provide auxiliary coefficients for reducing, by a very considerable factor, the amount of work in applying (1). The principle involved is a generalization of a method originally given by W. J. Taylor[1], to the extent of going beyond the Lagrangian interpolation formula (the author's previous generalizations to unequally spaced points or to complex points were all for the Lagrangian formula).

Rewriting (1) in the form

$$(2) \qquad f(x) = \prod_{j=0}^{2n} \sin \tfrac{1}{2}(x - x_j) \sum_{i=0}^{2n} \frac{1}{\left[\prod_{j=0}^{2n}{}' \sin \frac{1}{2}(x_i - x_j)\right] \sin \frac{1}{2}(x - x_i)} f_i,$$

let $1/\prod'^{2n}_{j=0} \sin \frac{1}{2}(x_i - x_j)$ be denoted by $A_i^{(2n+1)}$. Then from the definition

$$(3) \qquad a_i^{(2n+1)} \equiv A_i^{(2n+1)} / \sin \tfrac{1}{2}(x - x_i),$$

it follows from the unique determination of the coefficients of a trigonometric sum of the n^{th} order which assumes $2n + 1$ preassigned values[2], when applied to the case $f_0 = f_1 = \cdots = f_{2n} = 1$, similar to ordinary Lagrangian coefficients, that

$$(4) \qquad 1 = \prod_{j=0}^{2n} \sin \tfrac{1}{2}(x - x_j) \sum_{i=0}^{2n} a_i^{(2n+1)}, \quad \text{from which}$$

$$(5) \qquad f(x) = \sum_{i=0}^{2n} a_i^{(2n+1)} f_i \Big/ \sum_{i=0}^{2n} a_i^{(2n+1)}.$$

The present tables give $A_i^{(2n+1)} \equiv A_{2n-i}^{(2n+1)}$ for equally spaced x_i's, for some of the more important cases likely to arise in practice. Unlike the corresponding

[1] "Method of Lagrangian Curvilinear Interpolation," *Jour. Res. Nat. Bur. Stand.*, vol. 35, Aug. 1945, RP 1667, pp. 151–155.

[2] For a proof of the statement that two trigonometric sums of the n^{th} order which coincide in value at $2n + 1$ distinct points of a period are identically equal, see Dunham Jackson, *The Theory of Approximation*, American Mathematical Society Colloquium Publications, vol. XI, 1930, Chapter IV, pp. 109–111.

quantities for Lagrangian interpolation, here separate sets of $A_i^{(2n+1)}$ are required for each interval in x. Coefficients $A_i^{(2n+1)}$ are given for the 3-, 5-, 7-, 9- and 11-point cases, all at intervals in x equal to 1.0, 0.5, 0.2, 0.1, 0.05, 0.02 and 0.01; also $A_i^{(2n+1)}$ are given for functions tabulated at $2n + 1$ equally spaced points over a range of π and $\pi/2$, for $2n + 1 = 3(2)11$, since those ranges, i.e. 180° and 90°, are important for many periodic functions. All the quantities $A_i^{(2n+1)}$ are given to eight significant figures.

It is worth noting that this scheme for saving labor in interpolation by Gauss's formula, would be applicable to any type of interpolation formula which is linear in the f_i's, where

a) the coefficient of f_i is of the form $\prod_{j=0}'^{m} \phi(x - x_j)/\prod_{j=0}'^{m} \phi(x_i - x_j)$, i.e. "separable" into products, and

b) the sum of the coefficients of f_i is 1, or even some fixed quantity $C(x_0, x_1, \cdots, x_m)$ which depends upon the x_i's but not upon x, so that from $A_i^{(m+1)} \equiv 1/\prod_{j=0}'^{m} \phi(x_i - x_j)$, one defines

$$a_i^{(m+1)} \equiv A_i^{(m+1)}/\phi(x - x_i), \text{ from which} \tag{6}$$

$$f(x) = \frac{C(x_0, x_1, \cdots x_m)}{\sum_{i=0}^{m} a_i^{(m+1)}} \sum_{i=0}^{m} a_i^{(m+1)} f_i . \tag{7}$$

It would be interesting to enumerate other functions $\phi(x)$ besides $\phi(x) \equiv x$ and $\phi(x) \equiv \sin \frac{1}{2} x$ which have properties a) and b) and which would be useful in interpolating for other special types of functions besides polynomials and trigonometric sums.

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TABLE OF COEFFICIENTS $A_i^{(2n+1)}$[1]

Interval = 1.0		
3-point: $A_0^{(3)} = A_2^{(3)} = 2.47878\ 97$	$A_1^{(3)} = -4.35068\ 53$	
5-point: $A_0^{(5)} = A_4^{(5)} = 2.73289\ 54$	$A_1^{(5)} = A_3^{(5)} = -5.18331\ 74$	$A_2^{(5)} = 6.14439\ 86$
7-point: $A_0^{(7)} = A_6^{(7)} = 3.23586\ 56(1)$	$A_1^{(7)} = A_5^{(7)} = -9.52484\ 46$	$A_2^{(7)} = A_4^{(7)} = 6.77427\ 30$
$A_3^{(7)} = -6.17529\ 82$		
9-point: $A_0^{(9)} = A_8^{(9)} = 1.21890\ 29(2)$	$A_1^{(9)} = A_7^{(9)} = 1.92411\ 26(2)$	$A_2^{(9)} = A_6^{(9)} = 8.02103\ 04\ (1)$
$A_3^{(9)} = A_5^{(9)} = -1.13477\ 05(1)$	$A_4^{(9)} = 7.46871\ 71$	
11-point: $A_0^{(11)} = A_{10}^{(11)} = 1.30033\ 32(2)$	$A_1^{(11)} = A_9^{(11)} = 2.60086\ 49(2)$	$A_2^{(11)} = A_8^{(11)} = 3.02140\ 39(2)$
$A_3^{(11)} = A_7^{(11)} = 2.29234\ 84(2)$	$A_4^{(11)} = A_6^{(11)} = 8.84328\ 21(1)$	$A_5^{(11)} = -2.08525\ 00(1)$
Interval = 0.5		
3-point: $A_0^{(3)} = A_2^{(3)} = 8.43086\ 61$	$A_1^{(3)} = -1.63375\ 42(1)$	
5-point: $A_0^{(5)} = A_4^{(5)} = 1.46986\ 95(1)$	$A_1^{(5)} = A_3^{(5)} = -4.99932\ 38(1)$	$A_2^{(5)} = 7.10795\ 03(1)$
7-point: $A_0^{(7)} = A_6^{(7)} = 1.55277\ 62(1)$	$A_1^{(7)} = A_5^{(7)} = -6.26055\ 66(1)$	$A_2^{(7)} = A_4^{(7)} = 1.23922\ 73(2)$
$A_3^{(7)} = -1.52980\ 41(2)$		
9-point: $A_0^{(9)} = A_8^{(9)} = 1.73545\ 76(1)$	$A_1^{(9)} = A_7^{(9)} = -6.37842\ 32(1)$	$A_2^{(9)} = A_6^{(9)} = 1.30912\ 48(2)$
$A_3^{(9)} = A_5^{(9)} = -1.91574\ 41(2)$	$A_4^{(9)} = 2.16051\ 62(2)$	
11-point: $A_0^{(11)} = A_{10}^{(11)} = 3.72691\ 61(1)$	$A_1^{(11)} = A_9^{(11)} = -9.01543\ 97(1)$	$A_2^{(11)} = A_8^{(11)} = 1.46314\ 11(2)$
$A_3^{(11)} = A_7^{(11)} = -1.95181\ 16(2)$	$A_4^{(11)} = A_6^{(11)} = 2.28237\ 83(2)$	$A_5^{(11)} = -2.39904\ 94(2)$
Interval = 0.2		
3-point: $A_0^{(3)} = A_2^{(3)} = 5.04188\ 85(1)$	$A_1^{(3)} = -1.00334\ 00(2)$	
5-point: $A_0^{(5)} = A_4^{(5)} = 4.38116\ 54(2)$	$A_1^{(5)} = A_3^{(5)} = -1.70895\ 30(3)$	$A_2^{(5)} = 2.54206\ 40(3)$
7-point: $A_0^{(7)} = A_6^{(7)} = 1.61843\ 38(3)$	$A_1^{(7)} = A_5^{(7)} = -9.15361\ 31(3)$	$A_2^{(7)} = A_4^{(7)} = 2.20893\ 48(4)$
$A_3^{(7)} = -2.91079\ 83(4)$		
9-point: $A_0^{(9)} = A_8^{(9)} = 3.50209\ 17(3)$	$A_1^{(9)} = A_7^{(9)} = -2.51643\ 87(4)$	$A_2^{(9)} = A_6^{(9)} = 8.15996\ 28(4)$
$A_3^{(9)} = A_5^{(9)} = -1.55910\ 20(5)$	$A_4^{(9)} = 1.91946\ 10(5)$	
11-point: $A_0^{(11)} = A_{10}^{(11)} = 5.31306\ 73(3)$	$A_1^{(11)} = A_9^{(11)} = -4.47825\ 20(4)$	$A_2^{(11)} = A_8^{(11)} = 1.76571\ 56(5)$
$A_3^{(11)} = A_7^{(11)} = -4.28615\ 97(5)$	$A_4^{(11)} = A_6^{(11)} = 7.09062\ 63(5)$	$A_5^{(11)} = -8.35097\ 09\ (5)$

[1] The number N in parentheses after an entry denotes the factor 10^N.

Interval = 0.1

3-point: $A_0^{(3)} = A_2^{(3)} = 2.00417\ 22(2)$	$A_1^{(3)} = -4.00333\ 50(2)$	
5-point: $A_0^{(5)} = A_4^{(5)} = 6.75060\ 62(3)$	$A_1^{(5)} = A_3^{(5)} = -2.68339\ 48(4)$	$A_2^{(5)} = 4.01670\ 62(4)$
7-point: $A_0^{(7)} = A_6^{(7)} = 9.23312\ 99(4)$	$A_1^{(7)} = A_5^{(7)} = -5.45942\ 74(5)$	$A_2^{(7)} = A_4^{(7)} = 1.35293\ 77(6)$
$A_3^{(7)} = -1.79865\ 22(6)$		
9-point: $A_0^{(9)} = A_8^{(9)} = 6.91461\ 23(5)$	$A_1^{(9)} = A_7^{(9)} = -5.38750\ 82(6)$	$A_2^{(9)} = A_6^{(9)} = 1.85047\ 82(7)$
$A_3^{(9)} = A_5^{(9)} = -3.65939\ 87(7)$	$A_4^{(9)} = 4.55706\ 84(7)$	
11-point: $A_0^{(11)} = A_{10}^{(11)} = 3.31582\ 67(6)$	$A_1^{(11)} = A_9^{(11)} = -3.18070\ 91(7)$	$A_2^{(11)} = A_8^{(11)} = 1.38580\ 74(8)$
$A_3^{(11)} = A_7^{(11)} = -3.61125\ 23(8)$	$A_4^{(11)} = A_6^{(11)} = 6.23292\ 23(8)$	$A_5^{(11)} = -7.44512\ 94(8)$

Interval = 0.05

3-point: $A_0^{(3)} = A_2^{(3)} = 8.00416\ 81(2)$	$A_1^{(3)} = -1.60033\ 34(3)$	
5-point: $A_0^{(5)} = A_4^{(5)} = 1.07000\ 60(5)$	$A_1^{(5)} = A_3^{(5)} = -4.27333\ 95(5)$	$A_2^{(5)} = 6.40667\ 06(5)$
7-point: $A_0^{(7)} = A_6^{(7)} = 5.74309\ 96(6)$	$A_1^{(7)} = A_5^{(7)} = -3.43330\ 99(7)$	$A_2^{(7)} = A_4^{(7)} = 8.56450\ 81(7)$
$A_3^{(7)} = -1.14110\ 16(8)$		
9-point: $A_0^{(9)} = A_8^{(9)} = 1.66033\ 78(8)$	$A_1^{(9)} = A_7^{(9)} = -1.31957\ 02(9)$	$A_2^{(9)} = A_6^{(9)} = 4.50687\ 34(9)$
$A_3^{(9)} = A_5^{(9)} = -9.16790\ 16(9)$	$A_4^{(9)} = 1.14491\ 29(10)$	
11-point: $A_0^{(11)} = A_{10}^{(11)} = 3.00800\ 01(9)$	$A_1^{(11)} = A_9^{(11)} = -2.97707\ 47(10)$	$A_2^{(11)} = A_8^{(11)} = 1.32896\ 23(11)$
$A_3^{(11)} = A_7^{(11)} = -3.52362\ 30(11)$	$A_4^{(11)} = A_6^{(11)} = 6.14515\ 12(11)$	$A_5^{(11)} = -7.36572\ 60(11)$

Interval = 0.02

3-point: $A_0^{(3)} = A_2^{(3)} = 5.00041\ 67(3)$	$A_1^{(3)} = -1.00003\ 33(4)$	
5-point: $A_0^{(5)} = A_4^{(5)} = 4.16875\ 06(6)$	$A_1^{(5)} = A_3^{(5)} = -1.66708\ 34(7)$	$A_2^{(5)} = 2.50041\ 67(7)$
7-point: $A_0^{(7)} = A_6^{(7)} = 1.39099\ 71(9)$	$A_1^{(7)} = A_5^{(7)} = -8.34111\ 52(9)$	$A_2^{(7)} = A_4^{(7)} = 2.08454\ 90(10)$
$A_3^{(7)} = -2.77907\ 44(10)$		
9-point: $A_0^{(9)} = A_8^{(9)} = 2.48860\ 68(11)$	$A_1^{(9)} = A_7^{(9)} = -1.98879\ 57(12)$	$A_2^{(9)} = A_6^{(9)} = 6.95556\ 53(12)$
$A_3^{(9)} = A_5^{(9)} = -1.39048\ 71(13)$	$A_4^{(9)} = 1.73784\ 82(13)$	
11-point: $A_0^{(11)} = A_{10}^{(11)} = 2.77347\ 53(13)$	$A_1^{(11)} = A_9^{(11)} = -2.76890\ 13(14)$	$A_2^{(11)} = A_8^{(11)} = 1.24440\ 71(15)$
$A_3^{(11)} = A_7^{(11)} = -3.31537\ 77(15)$	$A_4^{(11)} = A_6^{(11)} = 5.79872\ 02(15)$	$A_5^{(11)} = -6.95718\ 84(15)$

TABLE OF COEFFICIENTS $A_i^{(2n+1)}$—(*Continued*)

Interval = 0.01

3-point: $A_0^{(3)} = A_2^{(3)} = 2.00004\ 17(4)$	$A_1^{(3)} = -4.00003\ 33(4)$	
5-point: $A_0^{(5)} = A_4^{(5)} = 6.66750\ 01(7)$	$A_1^{(5)} = A_3^{(5)} = -2.66683\ 33(8)$	$A_2^{(5)} = 4.00016\ 67(8)$
7-point: $A_0^{(7)} = A_6^{(7)} = 8.89226\ 00(10)$	$A_1^{(7)} = A_5^{(7)} = -5.33457\ 79(11)$	$A_2^{(7)} = A_4^{(7)} = 1.33352\ 78(12)$
$A_3^{(7)} = -1.77798\ 52(12)$		
9-point: $A_0^{(9)} = A_8^{(9)} = 6.35460\ 57(13)$	$A_1^{(9)} = A_7^{(9)} = -5.08235\ 02(14)$	$A_2^{(9)} = A_6^{(9)} = 1.77848\ 90(15)$
$A_3^{(9)} = A_5^{(9)} = -3.55657\ 79(15)$	$A_4^{(9)} = 4.44555\ 57(15)$	
11-point: $A_0^{(11)} = A_{10}^{(11)} = 2.82640\ 01(16)$	$A_1^{(11)} = A_9^{(11)} = -2.82523\ 44(17)$	$A_2^{(11)} = A_8^{(11)} = 1.27094\ 76(18)$
$A_3^{(11)} = A_7^{(11)} = -3.38841\ 70(18)$	$A_4^{(11)} = A_6^{(11)} = 5.92891\ 44(18)$	$A_5^{(11)} = -7.11437\ 12(18)$

Range = $\pi = 180°$

3-point: $A_0^{(3)} = A_2^{(3)} = 1.41421\ 36$	$A_1^{(3)} = -2.00000\ 00$	
5-point: $A_0^{(5)} = A_4^{(5)} = 4.00000\ 00$	$A_1^{(5)} = A_3^{(5)} = -1.04525\ 04(1)$	$A_2^{(5)} = 1.36568\ 54(1)$
7-point: $A_0^{(7)} = A_6^{(7)} = 1.30639\ 45(1)$	$A_1^{(7)} = A_5^{(7)} = -5.04752\ 09(1)$	$A_3^{(7)} = A_4^{(7)} = 9.75106\ 15(1)$
$A_3^{(7)} = -1.19425\ 63(2)$		
9-point: $A_0^{(9)} = A_8^{(9)} = 4.52548\ 34(1)$	$A_1^{(9)} = A_7^{(9)} = -2.31968\ 63(2)$	$A_2^{(9)} = A_6^{(9)} = 5.94515\ 98(2)$
$A_3^{(9)} = A_5^{(9)} = -9.88643\ 94(2)$	$A_4^{(9)} = 1.16252\ 23(3)$	
11-point: $A_0^{(11)} = A_{10}^{(11)} = 1.61908\ 62(2)$	$A_1^{(11)} = A_9^{(11)} = -1.03499\ 33(3)$	$A_2^{(11)} = A_8^{(11)} = 3.30807\ 30(3)$
$A_3^{(11)} = A_7^{(11)} = -6.93002\ 25(3)$	$A_4^{(11)} = A_6^{(11)} = 1.05050\ 19(4)$	$A_5^{(11)} = -1.20190\ 32(4)$

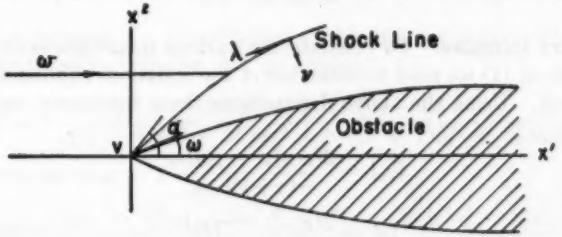
Range = $\pi/2 = 90°$

3-point: $A_0^{(3)} = A_2^{(3)} = 3.69551\ 81$	$A_1^{(3)} = -6.82842\ 71$	
5-point: $A_0^{(5)} = A_4^{(5)} = 3.40957\ 81(1)$	$A_1^{(5)} = A_3^{(5)} = -1.23580\ 49(2)$	$A_2^{(5)} = 1.79411\ 07(2)$
7-point: $A_0^{(7)} = A_6^{(7)} = 3.59388\ 61(2)$	$A_1^{(7)} = A_5^{(7)} = -1.94693\ 58(3)$	$A_3^{(7)} = A_4^{(7)} = 4.57933\ 63(3)$
$A_3^{(7)} = -5.98319\ 12(3)$		
9-point: $A_0^{(9)} = A_8^{(9)} = 4.00709\ 24(3)$	$A_1^{(9)} = A_7^{(9)} = -2.89076\ 19(4)$	$A_2^{(9)} = A_6^{(9)} = 9.40015\ 86(4)$
$A_3^{(9)} = A_5^{(9)} = -1.79907\ 82(5)$	$A_4^{(9)} = 2.21613\ 88(5)$	
11-point: $A_0^{(11)} = A_{10}^{(11)} = 4.60932\ 16(4)$	$A_1^{(11)} = A_9^{(11)} = -4.15411\ 69(5)$	$A_2^{(11)} = A_8^{(11)} = 1.72460\ 92(6)$
$A_3^{(11)} = A_7^{(11)} = -4.34234\ 30(6)$	$A_4^{(11)} = A_6^{(11)} = 7.34221\ 10(6)$	$A_5^{(11)} = -8.71031\ 71(6)$

CALCULATION OF THE CURVATURES OF ATTACHED SHOCK WAVES*

By T. Y. Thomas

1. Introduction. Consider a plane uniform flow of a gas for which the pressure, density and velocity vector are constant. Denote the pressure of this flow by p_1, the density by ρ_1 and let w be the magnitude of the velocity. The Mach number M of the flow is defined by $M = w/c_1$ where c_1 is the velocity of sound in the gas and is given by $c_1 = \gamma p_1/\rho_1$; here γ is the ratio of the two specific heats c_p and c_v, i.e. $\gamma = c_p/c_v$. Now place a pointed cylindrical obstacle in the flow so that the axis of the cylinder is perpendicular to the direction of flow; the flow past the obstacle will then be a plane two dimensional flow. To treat the problem we choose rectangular axes x^1 and x^2 in the plane of the flow, with origin at the vertex V of the obstacle, and such that the direction of the flow



coincides with the direction of the positive x^1 axis as shown in the figure. It is assumed that the incident flow is supersonic, i.e. $M > 1$, and that the conditions are otherwise such that a shock wave, or *shock line*, will originate from the vertex of the obstacle. In the figure we have shown the upper half of this wave which will furnish the basis of our consideration. The angle of inclination of the obstacle at its vertex V is denoted by ω and the corresponding angle for the shock line by α. Unit normal and tangent vectors to the shock line are denoted by ν and λ respectively; these vectors are directed as shown in the figure so that λ, ν have the same orientation as the coordinate axes. We suppose that the shock line and the flow behind this line are steady in character.

In a recent paper *On Curved Shock Waves*, Journal of Mathematics and Physics, vol. 26, 1947, pp. 62–68, we derived a formula relating the curvature κ of the shock line to the curvature K of the stream line immediately behind the shock line; this formula applies to the case of steady, rotational and non-adiabatic plane flow of an ideal gas, behind the shock line, with viscosity and thermal conductivity neglected. In the formula the curvature K is positive for stream lines which are concave upwards while the curvature κ of the shock line is negative under this condition. Hence if we replace κ by $-k$ both curvatures K and k

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will be positive for curves which are concave upwards and negative for curves which are concave downwards. For uniform flow before the shock line, which we assume, the modified formula, involving k, can be written in the form

$$(1)\qquad \frac{K}{k} = \frac{\left(\frac{\rho}{\rho_1} + \frac{\rho_1}{\rho} + \frac{6-2\gamma}{\gamma+1}\right)\left(\frac{u_t}{v}\right)^2 - \left(\frac{\rho}{\rho_1} - 1\right)\left(\frac{u_t}{v}\right)}{(u_n^2/c^2) - 1} + \frac{4}{\gamma+1}\left(\frac{u_t}{v}\right).$$

Here ρ denotes the density immediately behind the shock line. Also v is the magnitude of the velocity behind the shock; u_t is the tangential component of the velocity along the shock line; u_n is the component of the velocity behind the shock along the normal to the shock line; finally c is the velocity of sound behind the shock line and is given by $c^2 = \gamma p/\rho$ where p is the pressure and γ is the above ratio c_p/c_v. It is assumed that γ is constant throughout the flow.

In the following we shall be concerned primarily with the calculation of the ratio K/k at the vertex of the obstacle where the angle of inclination ω and the curvature K are specified by the boundary conditions of the problem.

2. Auxiliary formulas. To evaluate the various quantities occurring in the right member of (1) we need to make use of the equations expressing the conditions of shock. Using the above designations these conditions can be written in the form

$$(2)\qquad [u_\alpha] = \frac{2(\rho_1 u_{1n}^2 - \gamma p_1)\nu_\alpha}{-(\gamma+1)\rho_1 u_{1n}},$$

$$(3)\qquad [p] = \frac{2(\rho_1 u_{1n}^2 - \gamma p_1)}{\gamma+1},$$

$$(4)\qquad [\rho] = \frac{2\rho_1(\rho_1 u_{1n}^2 - \gamma p_1)}{2\gamma p_1 + (\gamma-1)\rho_1 u_{1n}^2},$$

where the quantities $[u_\alpha]$, $[p]$ and $[\rho]$ are the increments in the components of velocity, the pressure and the density respectively in the transition of the gas particles through the shock wave, e.g. $[u_\alpha] = u_\alpha - u_{1\alpha}$ with u_α the velocity components immediately behind the shock and $u_{1\alpha}$ the velocity components of the incident flow. Also ν_α are the components of the unit normal vector to the shock line directed into the region behind this line (as shown in the figure); correspondingly we shall denote by λ_α the components of the above unit tangent vector to the shock line. The quantity u_{1n} denotes the normal component of the incident flow and is constant only in the special case where the shock line is a straight line.

Now $u_{1n} = w\nu_1$. Hence, from (2), we have

$$u_1 = w - \frac{2(\rho_1 u_{1n}^2 - \gamma p_1)\nu_1}{(\gamma+1)\rho_1 u_{1n}} = w - \frac{2(u_{1n}^2 - c_1^2)\nu_1}{(\gamma+1)u_{1n}},$$

$$= \frac{(\gamma+1)wu_{1n} - 2u_{1n}^2\nu_1 + 2c_1^2\nu_1}{(\gamma+1)u_{1n}} = \frac{(\gamma+1)w^2\nu_1 - 2w^2\nu_1^3 + 2c_1\nu_1}{(\gamma+1)w\nu_1},$$

$$= \frac{(\gamma+1)w^2 - 2w^2\nu_1^2 + 2c_1^2}{(\gamma+1)w} = \frac{[(\gamma+1)M^2 - 2M^2\nu_1^2 + 2]c_1}{(\gamma+1)M}.$$

Also,

$$u_2 = \frac{-2(\rho_1 u_{1n}^2 - \gamma p_1)\nu_2}{(\gamma+1)\rho_1 u_{1n}} = \frac{-2(u_{1n}^2 - c_1^2)\nu_2}{(\gamma+1)u_{1n}},$$

$$= \frac{-2(w^2\nu_1^2 - c_1^2)}{(\gamma+1)w\nu_1} = \frac{-2(M^2\nu_1^2 - 1)\nu_2 c_1}{(\gamma+1)M\nu_1}.$$

But at the vertex V of the obstacle we have

$$\lambda_1 = \nu_2 = -\cos\alpha, \lambda_2 = -\nu_1 = -\sin\alpha.$$

Hence the above expressions for u_1 and u_2 become

$$u_1 = \frac{[(\gamma+1)M^2 - 2M^2\sin^2\alpha + 2]c_1}{(\gamma+1)M}, \tag{5}$$

$$u_2 = \frac{2(M^2\sin^2\alpha - 1)\cos\alpha\, c_1}{(\gamma+1)M\sin\alpha}. \tag{6}$$

Now $\tan\omega = u_2/u_1$ at V. Substituting in this the expressions (5) and (6) for u_1 and u_2 the resulting equation can be written

$$\tan\omega = \frac{2[(M^2-1)\tan^2\alpha - 1]}{[(\gamma-1)M^2+2]\tan^2\alpha + [(\gamma+1)M^2+2]\tan\alpha}. \tag{7}$$

From (3) we see that

$$[p] = \frac{2(u_{1n}^2 - c_1^2)\rho_1}{\gamma+1} = \frac{2(w^2\nu_1^2 - c_1^2)\rho_1}{\gamma+1} = \frac{2(M^2\nu_1^2-1)c_1^2\rho_1}{\gamma+1},$$

$$= \frac{2\gamma}{\gamma+1}\cdot\frac{(M^2\nu_1^2-1)c_1^2 p_1}{(\gamma p_1/\rho)} = \frac{2\gamma}{\gamma+1}(M^2\nu_1^2-1)p_1.$$

Dividing both members of this equation by p_1 we obtain

$$p/p_1 = [2\gamma/(\gamma+1)](M^2\sin^2\alpha - 1) + 1. \tag{8}$$

Now treat (4) in a similar manner, obtaining

$$\frac{[\rho]}{\rho_1} = \frac{2(u_{1n}^2 - c_1^2)}{2c_1^2 + (\gamma-1)u_{1n}^2} = \frac{2(M^2\nu_1^2-1)}{2+(\gamma-1)M^2\nu_1^2},$$

and hence,

$$\frac{\rho}{\rho_1} = \frac{2(M^2\sin^2\alpha - 1)}{(\gamma-1)M^2\sin^2\alpha + 2} + 1. \tag{9}$$

Now

$$c^2 = \frac{\gamma p}{\rho} = \frac{p}{p_1}\cdot\frac{\rho_1}{\rho}\cdot\frac{\gamma p_1}{\rho_1} = \frac{(p/p_1)c_1^2}{(\rho/\rho_1)}. \tag{10}$$

Hence the coefficient of c_1^2 in the right member of this equation is determined by the formulas (8) and (9). Also $u_n = u_\alpha\nu_\alpha$ and on expansion this gives

$$
\begin{aligned}
u_n &= u_1\nu_1 + u_2\nu_2 = -u_1\lambda_2 + u_2\lambda_1 \\
&= -u_1\lambda_1 \tan\alpha + u_1\lambda_1 \tan\omega \\
&= (\tan\omega - \tan\alpha)u_1\lambda_1 .
\end{aligned} \tag{11}
$$

But,

$$
\begin{aligned}
u_t &= u_a\lambda_a = u_1\lambda_1 + u_2\lambda_2 \\
&= u_1\lambda_1 + u_1 \tan\omega\,\lambda_1 \tan\alpha \\
&= (1 + \tan\omega \tan\alpha)\, u_1\lambda_1 .
\end{aligned} \tag{12}
$$

Hence combining (11) and (12) we have

$$
u_n = \frac{(\tan\omega - \tan\alpha)u_t}{1 + \tan\omega \tan\alpha} = \tan(\omega - \alpha)u_t . \tag{13}
$$

Since u_t is continuous across the shock line we also have $u_t = M\lambda_1 c_1 = -Mc_1 \cos\alpha$. Hence (13) becomes

$$
u_n = M \cos\alpha \tan(\alpha - \omega)c_1 . \tag{14}
$$

Then from (10) and (14) we have

$$
\frac{u_n^2}{c^2} = \frac{M^2 \cos^2\alpha \tan^2(\alpha - \omega)(\rho/\rho_1)}{p/p_1} \tag{15}
$$

For the determination of the right member of (1) we also need the ratio u_t/v. But from the above figure (see remarks in §4) we see immediately that this is given by

$$
u_t/v = -\cos(\alpha - \omega). \tag{16}
$$

3. Tables and curvature graphs. Those formulas from the preceding section which are needed directly in the calculation of the right member of equation (1) will be listed separately as follows

$$
\text{(A)} \qquad \tan\omega = \frac{2[(M^2 - 1)\tan^2\alpha - 1]}{[(\gamma - 1)M^2 + 2]\tan^3\alpha + [(\gamma + 1)M^2 + 2]\tan\alpha},
$$

$$
\text{(B)} \qquad \frac{p}{p_1} = \frac{2\gamma}{\gamma + 1}(M^2 \sin^2\alpha - 1) + 1,
$$

$$
\text{(C)} \qquad \frac{\rho}{\rho_1} = \frac{2(M^2 \sin^2\alpha - 1)}{(\gamma - 1)M^2 \sin^2\alpha + 2} + 1,
$$

$$
\text{(D)} \qquad \frac{u_n^2}{c^2} = \frac{M^2 \cos^2\alpha \tan^2(\alpha - \omega)(\rho/\rho_1)}{p/p_1},
$$

$$
\text{(E)} \qquad u_t/v = -\cos(\alpha - \omega).
$$

In the calculations we have taken $\gamma = 1.405$. Since the obstacle is assumed pointed at V the angle ω will be subjected to the restriction $0 \leqq \omega < 90°$. For

convenience in the calculations we take M and α as independent variables and consider as possible, *ab initio*, all values of α satisfying the inequality $0 \leqq \alpha \leqq 180°$. However since the wave is a compression shock, the condition $\rho/\rho_1 > 1$ must be satisfied and this further restricts the values which can be assumed by α and ω. In fact it follows from this condition that the first term in the right member of (C) must be positive and since the denominator of this term is necessarily positive the numerator must also be positive. Hence $\sin \alpha > 1/M$. This excludes the values $\alpha = 0°$ and $\alpha = 180°$. If α is assumed to lie in the second quadrant the denominator in the right member of (A) is negative; but

$$M^2 \sin^2 \alpha - 1 = [(M^2 - 1) \tan^2 \alpha - 1] \cos^2 \alpha > 0,$$

and hence the numerator of this member is positive whereas $\tan \omega \geqq 0$ from the inequality satisfied by ω. Hence α cannot be a second quadrant angle. Also $\sin \alpha = 1/M$ or $\alpha = 90°$ implies $\tan \omega = 0$ and conversely. *Hence in view of the condition* $\sin \alpha > 1/M$ *we must associate* $\omega = 0$ *with* $\alpha = 90°$ *only.*

Denoting by $\alpha_0(M)$ the value of α in the first quadrant such that $\sin \alpha = 1/M$ the following inequality is now seen to be satisfied by the angle α, namely

$$\alpha_0(M) < \alpha \leqq 90°.$$

In calculating the values of the ratio K/k and the values of the quantities which appear in the left members of the equations (A), $\cdots$, (E) we have selected values of M so that the values of $\alpha_0(M)$ are more or less equally spaced throughout the interval from $M = 1.05$ to $M = 4.45$. The results of these calculations are given in Tables I to XIII and the graphs of ω and the ratio K/k have been plotted as functions of α (for fixed M). The values of α in these tables have not been equally spaced but have been chosen in a way which appears to be expedient from the standpoint of the construction of the graphs.

4. Exceptional shock angles. From the graphs of the functions ω and K/k it is seen that certain values of the shock angle α merit special consideration. One of these is the value $\alpha = \alpha(M)$ at which ω assumes its maximum value $\omega(M)$. The equation determining $\alpha(M)$ is obtained in the usual manner by differentiation of (A) and is found to be

$$\tan^2 \alpha(M) = \frac{A + \sqrt{A^2 + 4BC}}{2B},$$

where,

$$A = [(\gamma + 1)M^2 + 2](M^2 - 1) + 3[(\gamma - 1)M^2 + 2],$$
$$B = [(\gamma - 1)M^2 + 2](M^2 - 1),$$
$$C = [(\gamma + 1)M^2 - 2].$$

Another exceptional value of the angle α is the value $\alpha = \alpha_s(M)$ at which the K/k curves cross the α axis. It is seen from the graphs that this value of α is, in each case, but slightly less than the value of α at which ω assumes its maximum. A value of α for which $K/k = 0$ will be called *singular*, more precisely

this value of α will be called *singular relative to the value of M in question.* The value of ω corresponding to a singular value of α will also be said to be *singular relative to M.* Thus the above values $\alpha_s(M)$ and the corresponding values of $\omega = \omega_s(M)$ are singular relative to M. Also we see from the graphs that $\alpha = 90°$ and the corresponding value $\omega = 0$ are singular for all values of M.

In Table XIV we have listed the values of $\alpha(M)$, $\omega(M)$, $\alpha_s(M)$ and $\omega_s(M)$ corresponding to the values of M which appear in Tables I to XIII. The values of $\alpha(M)$ and $\omega(M)$ in this table have been calculated while the values of $\alpha_s(M)$ and $\omega_s(M)$ have been read from the graphs of the functions K/k and ω. Finally we have constructed the graphs of the functions $\alpha(M)$, $\omega(M)$, $\alpha_s(M)$ and $\omega_s(M)$ from Table XIV.

TABLE I

$M = 1.05$ $\quad \alpha_0(M) = 72.250°$

α(deg.)	ω(deg.)	ρ/ρ_1	ρ_1/ρ	p/p_1	$-u_t/v$	u_n^2/c^2	K/k	k/K
73	.112	1.0068	.9932	1.0096	.2943	.9925	10.8212	.09241
73.5	.176	1.0112	.9889	1.0158	.2870	.9865	5.1101	.1957
74	.235	1.0156	.9846	1.0220	.2795	.9814	3.2050	.3120
75	.340	1.0237	.9768	1.0335	.2645	.9720	1.5344	.6517
76	.424	1.0314	.9696	1.0444	.2491	.9632	.7710	1.2970
77	.484	1.0386	.9628	1.0547	.2332	.9561	.3813	2.6226
78	.526	1.0452	.9568	1.0640	.2169	.9483	.1064	9.3985
79	.550	1.0512	.9513	1.0727	.2002	.9452	−.0326	−30.6748
80	.556	1.0569	.9462	1.0809	.1832	.9356	−.1486	−6.7295
82	.528	1.0666	.9376	1.0949	.1483	.9259	−.2334	−4.2845
84	.441	1.0740	.9311	1.1056	.1122	.9177	−.2303	−4.3422
86	.318	1.0795	.9264	1.1136	.07532	.9131	−.1777	−5.6275
88	.167	1.0827	.9236	1.1182	.03781	.9083	−.09500	−10.5263
90	.000	1.0838	.9227	1.1198	.0000	.9085	.0000	$-\infty$

TABLE II

$M = 1.08$ $\quad \alpha_0(M) = 67.8°$

α(deg.)	ω(deg.)	ρ/ρ_1	ρ_1/ρ	p/p_1	$-u_t/v$	u_n^2/c^2	K/k	k/K
70	.455	1.0248	.9758	1.0350	.3495	.9715	4.072	.2456
72	.759	1.0454	.9566	1.0644	.32155	.9486	1.335	.7491
74	.963	1.0639	.9399	1.0910	.2917	.9286	.4114	2.4307
76	1.074	1.0803	.9257	1.1147	.2631	.9121	.0123	81.300
78	1.098	1.0945	.9137	1.1354	.2267	.8976	−.2068	−4.8356
80	1.041	1.1067	.9036	1.1533	.19155	.8861	−.2922	−3.3422
82	.920	1.1168	.8954	1.1681	.1551	.8773	−.3041	−3.2884
84	.741	1.1245	.8893	1.1795	.11735	.8692	−.2656	−3.7651
86	.519	1.1302	.8848	1.1879	.0787	.8651	−.1948	−5.1335
88	.268	1.1335	.8822	1.1928	.03979	.8603	−.1027	−9.7371
90	.000	1.1346	.8814	1.1944	.0000	.8607	.0000	$-\infty$

TABLE III

$M = 1.12$ $\alpha_0(M) = 63.238°$

α(deg.)	ω(deg.)	ρ/ρ_1	ρ_1/ρ	p/p_1	$-u_t/v$	u_n^2/c^2	K/k	k/K
64	.250	1.0110	.9891	1.0155	.4423	.9870	21.0259	.04756
67	1.057	1.0517	.9508	1.0734	.4076	.9410	2.7848	.3591
70	1.644	1.0879	.9192	1.1257	.3688	.9010	.7482	1.3365
73	1.892	1.1194	.8933	1.1719	.32375	.8752	.0593	16.8634
76	1.946	1.1461	.8725	1.2115	.2747	.8506	−.2614	−3.8256
79	1.789	1.1676	.8565	1.2437	.2214	.8320	−.3727	−2.6831
82	1.450	1.1843	.8444	1.2690	.1642	.8189	−.3582	−2.7917
85	.970	1.1956	.8364	1.2861	.1040	.8109	−.2606	−3.8373
88	.402	1.2017	.8322	1.2955	.0420	.8049	−.1120	−8.9286
90	.000	1.2029	.8313	1.2972	.0000	.8039	.0000	$-\infty$

TABLE IV

$M = 1.18$ $\alpha_0(M) = 57.944°$

α(deg.)	ω(deg.)	ρ/ρ_1	ρ_1/ρ	p/p_1	$-u_t/v$	u_n^2/c^2	K/k	k/K
60	.900	1.0366	.9647	1.0518	.5135	.9578	9.378	.1066
61.5	1.467	1.0619	.9417	1.0880	.4995	.9307	4.7103	.2123
63	1.961	1.0861	.9207	1.1231	.4842	.9062	2.7802	.3597
64	2.249	1.1016	.9078	1.1458	.4733	.8913	2.0233	.4942
66	2.730	1.1311	.8841	1.1893	.4498	.8636	1.0488	.9535
69	3.212	1.1715	.8536	1.2496	.4101	.8295	.2616	3.8226
70	3.307	1.1838	.8447	1.2681	.3956	.8193	.0891	11.2233
71	3.375	1.1955	.8365	1.2860	.38065	.8097	−.0509	−19.6464
72	3.410	1.2068	.8286	1.3033	.3650	.8004	−.1662	−6.0168
75	3.334	1.2368	.8085	1.3495	.3145	.7784	−.3859	−2.5913
78	3.012	1.2615	.7927	1.3880	.2590	.7606	−.4684	−2.1349
81	2.468	1.2811	.7806	1.4188	.1988	.7479	−.4468	−2.2381
84	1.751	1.2949	.7723	1.4406	.1349	.7377	−.3443	−2.9044
87	.908	1.3032	.7673	1.4539	.06815	.7313	−.1862	−5.3706
90	.000	1.3061	.7656	1.4585	.0000	.7309	.0000	$-\infty$

TABLE V

$M = 1.25$ $\alpha_0(M) = 53.170°$

α(deg.)	ω(deg.)	p/p_1	ρ/ρ_1	ρ_1/ρ	$-u_1/v$	u_n^2/c^2	K/h	h/K
54	.510	1.0265	1.0188	.9815	.5950	.9777	29.925	.03342
56	1.584	1.0862	1.0606	.9429	.5819	.9319	8.1467	.1227
57	2.069	1.1157	1.0810	.9251	.5745	.9111	5.6283	.1777
58	2.521	1.1444	1.1007	.9085	.5667	.8919	4.1454	.2412
59	2.941	1.1730	1.1202	.8927	.55835	.8737	3.1435	.3181
60	3.326	1.2008	1.1389	.8780	.5494	.8568	2.4242	.4125
62	3.996	1.2547	1.1749	.8511	.5299	.8261	1.4540	.6878
66	4.913	1.3551	1.2404	.8062	.4866	.7753	.3994	2.5038
68	5.161	1.4011	1.2699	.7875	.4565	.7547	.0507	19.7239
70	5.266	1.4436	1.2968	.7711	.4268	.7371	−.1880	−5.3191
74	5.069	1.5187	1.3435	.7443	.3595	.7076	−.4779	−2.0925
76	4.768	1.5504	1.3630	.7337	.3217	.6961	−.5443	−1.8372
78	4.348	1.5781	1.3798	.7247	.2815	.6862	−.5651	−1.7696
80	3.814	1.6021	1.3943	.7172	.2387	.6779	−.5440	−1.8382
82	3.179	1.6219	1.4062	.7111	.1938	.6723	−.4860	−2.0576
86	1.678	1.6489	1.4221	.7032	.0989	.6642	−.2789	−3.5855
90	0.000	1.6572	1.4273	.7006	.0000	.6606	.0000	$-\infty$

TABLE VI

$M = 1.35$ $\alpha_0(M) = 47.8°$

α(deg.)	ω(deg.)	p/p_1	ρ/ρ_1	ρ_1/ρ	$-u_1/v$	u_n^2/c^2	K/h	h/K
48	.154	1.0075	1.0053	.9947	.6711	.9935	153.03	.026535
52	2.857	1.1538	1.1071	.9033	.6542	.8862	6.5059	.1537
54	4.011	1.2253	1.1553	.8656	.6429	.8425	3.9440	.2535
56	5.022	1.2949	1.2013	.8324	.6296	.8051	2.6060	.3837
60	6.622	1.4287	1.2874	.7768	.5966	.7433	1.1437	.8744
64	7.633	1.5517	1.3638	.7332	.5539	.6957	.3279	3.0497
66	7.906	1.6086	1.3982	.7152	.5285	.6759	.0383	26.1097
68	8.022	1.6622	1.4303	.6992	.5004	.6589	−.1941	−5.1520
72	7.772	1.7578	1.4863	.6728	.4348	.6311	−.5196	−1.9246
74	7.411	1.7994	1.5102	.6622	.3973	.6196	−.6169	−1.6210
76	6.894	1.8365	1.5313	.6530	.3566	.6105	−.6709	−1.4905
80	5.430	1.8967	1.5652	.6389	.2660	.5950	−.6495	−1.5396
84	3.474	1.9376	1.5879	.6298	.1646	.5857	−.4692	−2.1313
88	1.199	1.9584	1.5994	.6252	.0558	.58058	−.1719	−5.8173
90	0.000	1.9610	1.6008	.6247	.0000	.58057	.0000	$-\infty$

TABLE VII

$M = 1.47$ $\alpha_0(M) = 42.869°$

α(deg.)	ω(deg.)	p/p_1	ρ/ρ_1	ρ_1/ρ	$-u_1/v$	u_n^2/c^2	K/k	k/K
45	1.831	1.0940	1.0660	.9381	.7294	.9263	15.6729	.06380
46	2.633	1.1379	1.0962	.9122	.7270	.8965	10.4943	.09529
50	5.513	1.3131	1.2132	.8243	.7135	.7958	4.0518	.2468
53	7.317	1.4419	1.2957	.7718	.6986	.7376	2.4753	.4040
55	8.344	1.5260	1.3480	.7418	.6864	.7049	1.7927	.5578
57	9.222	1.6076	1.3976	.7155	.6720	.6767	1.2847	.7784
60	10.272	1.7252	1.4673	.6815	.6464	.6402	.6942	1.4405
62	10.767	1.7997	1.5103	.6621	.6262	.6198	.3783	2.6434
64	11.106	1.8712	1.5509	.6448	.6033	.6015	.1035	9.6618
65	11.211	1.9055	1.5701	.6369	.5907	.5931	−.0206	−48.544
70	11.061	2.0610	1.6550	.6042	.5159	.5595	−.5075	−1.9704
75	9.767	2.1872	1.7215	.5809	.4189	.5351	−.77285	−1.2939
80	7.328	2.2801	1.7692	.5652	.2978	.5191	−.7706	−1.2977
85	3.939	2.3423	1.8005	.5554	.1554	.5105	−.4844	−2.0644
90	0.000	2.3564	1.8075	.5533	.0000	.5073	.0000	$-\infty$

TABLE VIII

$M = 1.63$ $\alpha_0(M) = 37.843°$

α(deg.)	ω(deg.)	p/p_1	ρ/ρ_1	ρ_1/ρ	$-u_1/v$	u_n^2/c^2	K/k	k/K
40	2.154	1.1143	1.0800	.9259	.7896	.9122	14.8145	.0675
45	6.539	1.3838	1.2588	.7944	.7831	.7625	4.6781	.2138
50	10.094	1.6533	1.4249	.7018	.7671	.6617	2.3683	.4222
55	12.800	1.9149	1.5753	.6348	.7408	.5912	1.2420	.8052
60	14.589	2.1599	1.7073	.5857	.7020	.5403	.4759	2.1013
62	15.016	2.2515	1.7546	.5699	.6822	.5241	.2195	4.5558
65	15.322	2.3815	1.8200	.5495	.6471	.5035	−.1277	−7.8309
70	14.842	2.5727	1.9125	.5229	.5313	.4770	−.7317	−1.3667
75	13.000	2.7279	1.9846	.5039	.4695	.4580	−.9054	−1.1045
80	9.750	2.8422	2.0361	.4911	.3379	.4450	−.9249	−1.0812
85	5.250	2.9185	2.0697	.4832	.1779	.4381	−.5958	−1.6784
90	0.000	2.9359	2.0773	.4814	.0000	.4356	.0000	$-\infty$

TABLE IX

$M = 1.83$ $\alpha_0(M) = 33.120°$

α(deg.)	ω(deg.)	p/p_1	ρ/ρ_1	ρ_1/ρ	$-u_1/v$	u_n^2/c^2	K/k	k/K
35	2.120	1.1189	1.0832	.9232	.8398	.9090	19.5274	.0512
40	7.153	1.4484	1.2998	.7693	.8401	.7350	5.2491	.1905
45	11.378	1.7880	1.5036	.6651	.8327	.6225	2.8503	.3508
50	14.816	2.1276	1.6904	.5916	.8173	.5464	1.7392	.5750
55	17.437	2.4575	1.8573	.5384	.7926	.4925	.9944	1.0056
60	19.137	2.7663	2.0021	.4995	.7562	.4535	.3844	2.6015
62	19.520	2.8817	2.0535	.4870	.7375	.4410	.1580	6.3291
65	19.732	3.0456	2.1244	.4707	.7038	.4252	−.1692	−5.9102
70	18.989	3.2867	2.2241	.4496	.6292	.4046	−.6666	−1.5002
75	16.658	3.4823	2.3012	.4346	.52485	.3898	−1.0294	−.9714
78	14.411	3.5750	2.3367	.4280	.4448	.3834	−1.1203	−.8926
80	12.561	3.6263	2.3560	.4244	.3837	.3798	−1.1030	−.9066
85	6.806	3.7147	2.3888	.4186	.2046	.3748	−.7381	−1.3548
90	0.000	3.7445	2.3997	.4167	.0000	.3727	.0000	$-\infty$

TABLE X

$M = 2.12$ $\alpha_0(M) = 28.144°$

α(deg.)	ω(deg.)	p/p_1	ρ/ρ_1	ρ_1/ρ	$-u_1/v$	u_n^2/c^2	K/k	k/K
30	2.322	1.1444	1.1007	.9085	.8856	.8919	19.1349	.05226
35	7.900	1.5593	1.3684	.7308	.8902	.6929	5.3201	.1880
40	12.658	2.0014	1.6229	.6162	.8883	.5716	3.0646	.3263
45	16.700	2.4572	1.8572	.5384	.8805	.4924	2.0537	.4859
50	20.042	2.9130	2.0673	.4837	.8664	.4378	1.4000	.7143
55	22.610	3.3557	2.2517	.4441	.8444	.3994	.8682	1.1518
60	24.286	3.7700	2.4090	.4151	.8119	.3706	.3612	2.7685
65	24.829	4.1450	2.5403	.3937	.7641	.3507	−.1563	−6.3980
70	23.924	4.4685	2.6459	.3779	.6937	.3355	−.6868	−1.4560
75	21.155	4.7311	2.7269	.3667	.5900	.3250	−1.1457	−.8728
78	18.432	4.8555	2.7639	.3618	.5065	.3203	−1.3060	−.7657
80	16.147	4.9243	2.7840	.3592	.4407	.3179	−1.3247	−.7549
85	8.850	5.0429	2.8181	.3548	.2394	.3142	−.9430	−1.0604
90	0.000	5.0829	2.8294	.3534	.0000	.3125	.0000	$-\infty$

TABLE XI

$M = 2.56$ $\alpha_0(M) = 22.994°$

α(deg.)	ω(deg.)	p/p_1	ρ/ρ_1	ρ_1/ρ	$-u_t/v$	u_n^2/c^2	K/k	k/K
25	2.708	1.1992	1.1378	.8789	.92525	.8586	16.2871	.06140
30	8.683	1.7459	1.4794	.6759	.9316	.6343	4.9269	.2030
35	13.794	2.3508	1.8047	.5541	.9323	.5082	2.9958	.3338
45	22.110	3.6602	2.3686	.4222	.9213	.3781	1.6174	.6183
55	27.950	4.9703	2.7974	.3575	.8906	.3164	.8079	1.2378
60	29.683	5.5745	2.9622	.3376	.8632	.2977	.3894	2.5681
65	30.300	6.1212	3.0971	.3229	.8221	.2839	−.0922	−10.8460
70	29.387	6.5929	3.2040	.3121	.7591	.2740	−.6496	−1.5394
75	26.355	6.9758	3.2851	.3044	.6608	.2668	−1.2242	−.8169
80	20.510	7.2576	3.3418	.2992	.5077	.2619	−1.5742	−.6352
85	11.453	7.4306	3.3754	.2963	.2833	.2595	−1.2265	−.8153
90	00.000	7.4888	3.3865	.2953	.0000	.2584	0.0000	$-\infty$

TABLE XII

$M = 3.24$ $\omega_0(M) = 17.150°$

α(deg.)	ω(deg.)	p/p_1	ρ/ρ_1	ρ_1/ρ	$-u_t/v$	u_n^2/c^2	K/k	k/K
20	2.893	1.2666	1.1828	.8455	.9558	.8201	13.7221	.07288
25	9.072	2.0222	1.6341	.6120	.9616	.5676	4.3371	.2306
30	14.353	2.8979	2.0607	.4853	.9629	.4389	2.7400	.3650
40	23.186	4.8997	2.7769	.3601	.9572	.3186	1.6566	.6036
50	30.122	7.0289	3.2960	.3034	.9404	.2656	1.0736	.9314
55	32.748	8.0629	3.4911	.2864	.9255	.2503	.7822	1.2784
60	34.608	9.0306	3.6493	.2740	.9034	.2389	.4384	2.2810
60.432	34.727	9.1116	3.6616	.2731	.9011	.2379	.4074	2.4667
65	35.404	9.9064	3.7758	.2648	.8695	.2305	.0024	416.67
70	34.656	10.6620	3.8743	.2581	.8157	.2245	−.5589	−1.7860
75	31.608	11.2753	3.9478	.2533	.7227	.2200	−1.2582	−.7948
80	25.186	11.7266	3.9987	.2501	.5762	.2170	−1.8030	−.5546
82.5	20.365	11.8873	4.0162	.2490	.4674	.2161	−1.8476	−.5412
85	14.411	12.0038	4.0286	.2482	.3323	.2157	−1.5737	−.6354
87.5	7.494	12.0725	4.0359	.2478	.1735	.2149	−.9221	−1.0845
90	00.000	12.0970	4.0385	.2476	.0000	.2149	0.0000	$-\infty$

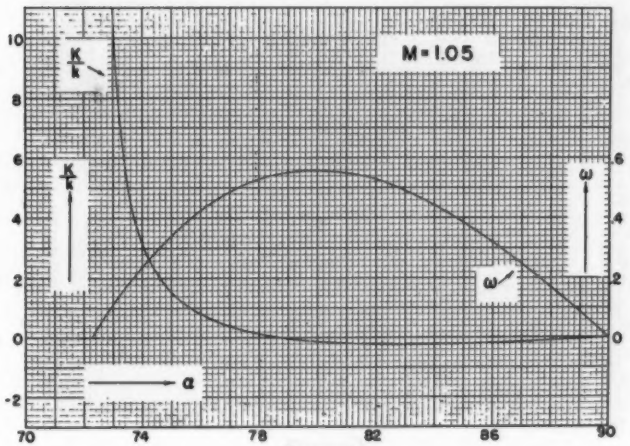
TABLE XIII

$M = 4.45 \qquad \alpha_0(M) = 12.982°$

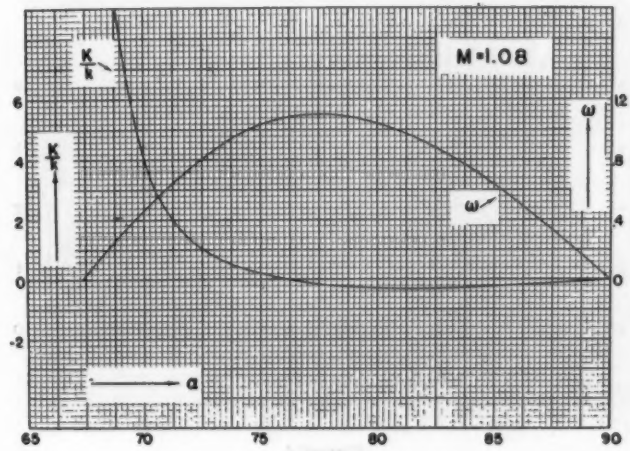
α(deg.)	ω(deg.)	p/p_1	ρ/ρ_1	ρ_1/ρ	$-u_1/v$	u_0^2/c^2	K/k	k/K
15	2.966	1.3814	1.2573	.7954	.9780	.7643	10.7175	.09331
20	9.094	2.5387	1.8964	.5273	.9820	.4850	3.5531	.2814
25	14.337	3.9639	2.4781	.4035	.9827	.3602	2.3370	.4279
35	23.286	7.6437	3.3779	.2960	.9792	.2524	1.5367	.6507
45	30.833	11.4002	3.9621	.2524	.9696	.2144	1.1603	.8618
50	34.042	13.4085	4.1673	.2400	.96145	.2079	.9901	1.0100
55	36.744	15.3591	4.3295	.2310	.9497	.1998	.7700	1.2987
60	38.761	17.1846	4.4565	.2224	.9321	.1939	.4878	2.0500
65	39.790	18.8366	4.5553	.2195	.9047	.1896	.1012	9.8814
70	39.317	20.2618	4.6305	.2160	.8600	.1864	−.4417	−2.2640
75	36.463	21.4186	4.6858	.2134	.7822	.1841	−1.1836	−.8449
80	29.791	22.2701	4.7236	.2117	.6400	.1824	−1.9718	−.5072
82	25.641	22.5223	4.7344	.2112	.55405	.1822	−2.1469	−.4658
85	17.547	22.7930	4.7457	.2107	.3835	.1819	−1.9605	−.5101
87.5	9.217	22.9226	4.7511	.2105	.2031	.1814	−1.2040	−.8306
89	3.742	22.9596	4.7526	.2104	.08265	.1825	−.5125	−1.9512
90	0.000	22.9688	4.7530	.2104	.00000	.1814	.0000	$-\infty$

TABLE XIV

M	$\alpha(M)$	$\omega(M)$	$\alpha_s(M)$	$\omega_s(M)$
1.05	79.937°	.557°	78.62°	.545°
1.08	77.558	1.099	76.20	1.08
1.12	75.207	1.955	73.40	1.92
1.18	72.653	3.416	70.40	3.35
1.25	70.534	5.271	68.50	5.21
1.35	68.563	8.028	66.35	7.96
1.47	66.870	11.289	64.60	11.18
1.63	65.635	15.328	63.50	15.20
1.83	64.890	19.732	63.50	19.60
2.12	64.582	24.833	63.60	24.80
2.56	64.782	30.300	64.20	30.20
3.24	65.399	35.408	65.20	35.40
4.45	66.246	39.838	66.00	39.70



GRAPH 1



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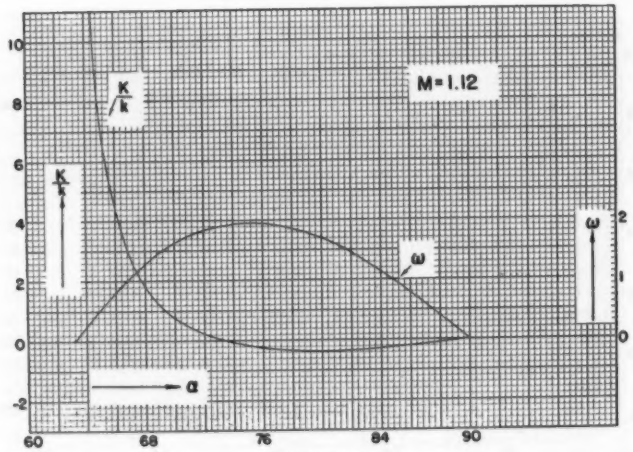

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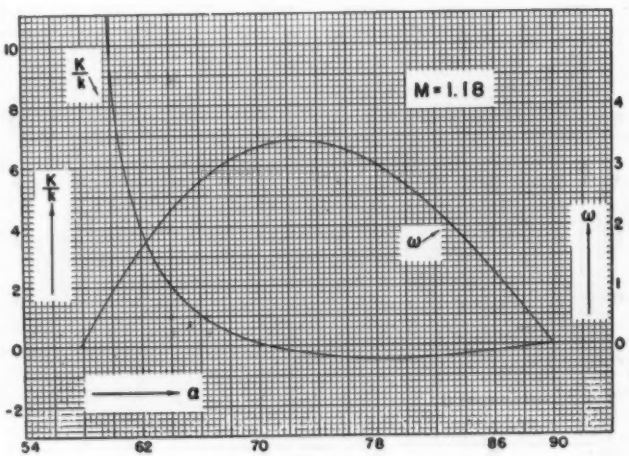

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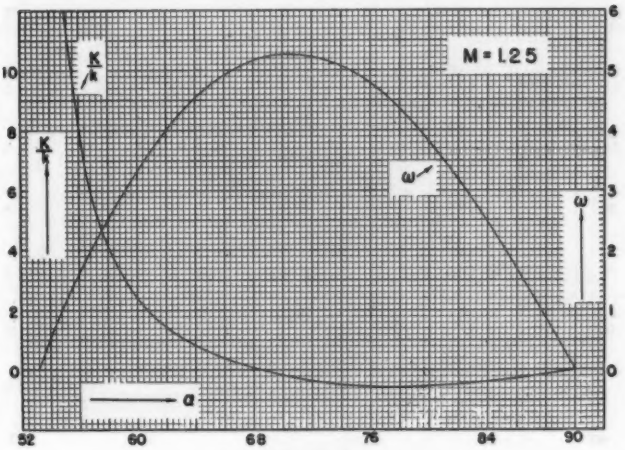

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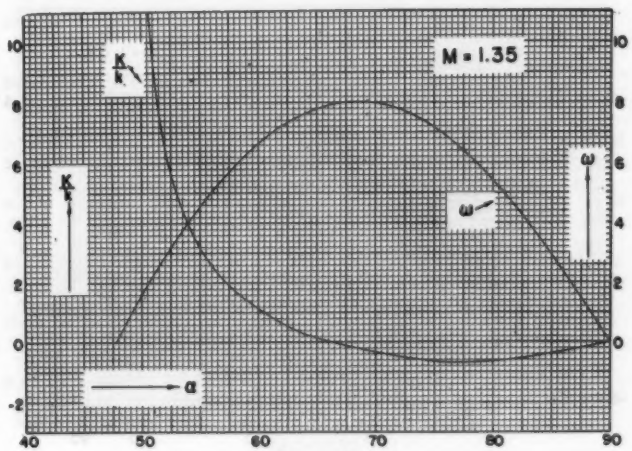

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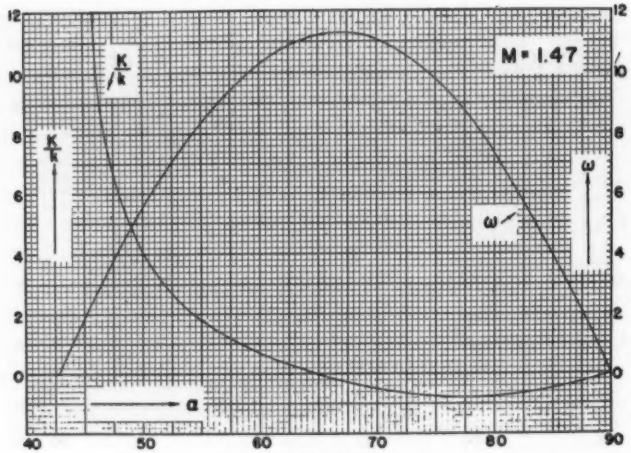

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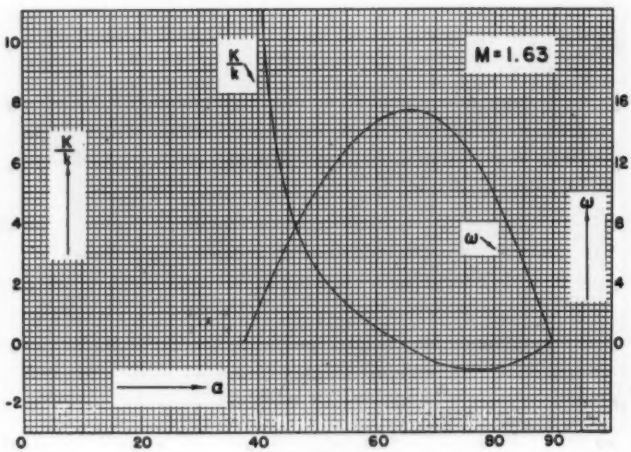

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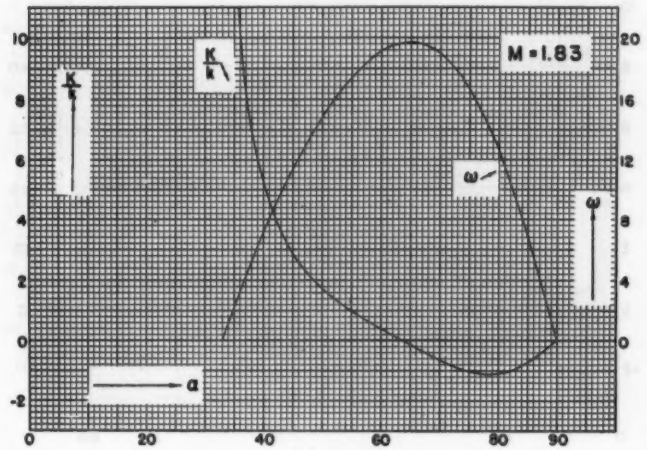

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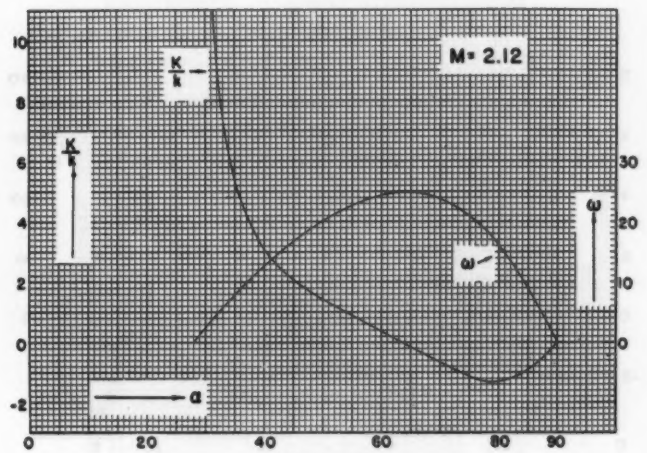

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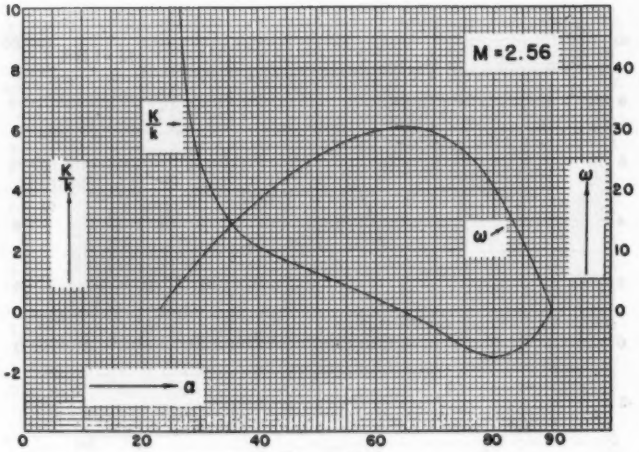

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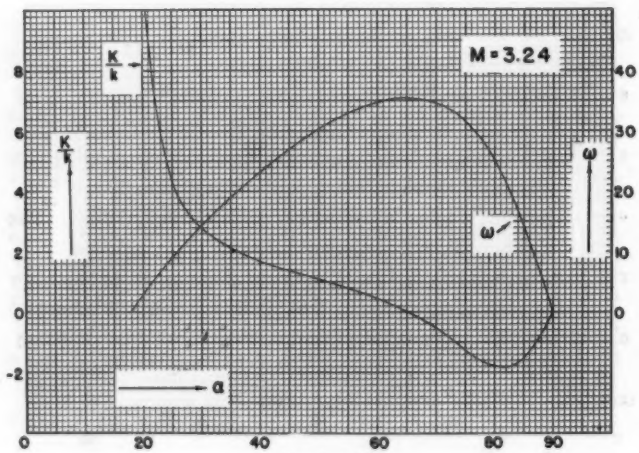

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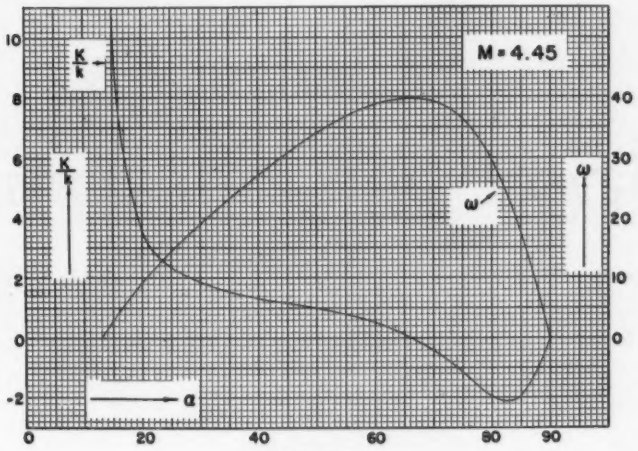

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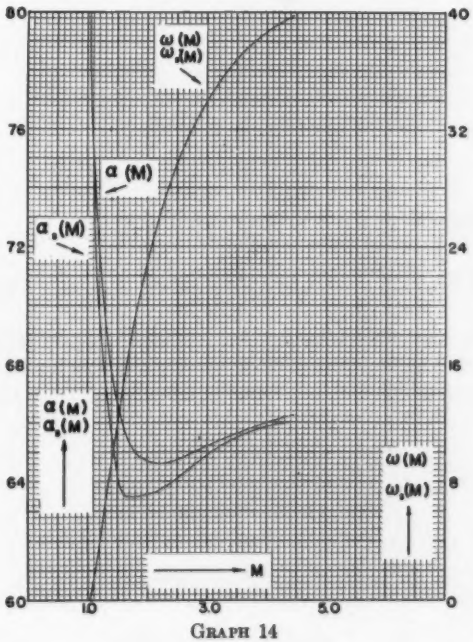

GRAPH 14

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